

1 Linear Algebra

Relevant definitions used throughout Analysis II.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

Def Euclidian Norm $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$

Used to generalize $|x|$ in many Analysis I definitions

Lem. Properties of $\|x\|$

- (i) $\|x\| \geq 0$
- (ii) $\|x\| \iff x = 0$
- (iii) $\|\alpha x\| = \alpha \cdot \|x\|$
- (iv) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

Def Trace $\text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$

2 Differential Equations

Def Differential Equation (DE)

Equation relating unknown f to derivatives $f^{(i)}$ at *same* x .

Def Ordinary Differential Equation (ODE)

DE s.t. $f : I \rightarrow \mathbb{R}$ is in one variable.

Def Partial Differential Equation (PDE)

DE s.t. $f : I^d \rightarrow \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def Order $\text{ord}(F) := \max_{i \geq 0} \{i \mid f^{(i)} \in F, f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \rightarrow \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$$\exists f : I \rightarrow \mathbb{R}$$

$$\text{s.t. } \forall x \in I : f'(x) = F(x, f(x)) \text{ and } f(x_0) = y_0$$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

2.1 Linear Differential Equations

Def Linear Differential Equation (LDE)

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

$I \subset \mathbb{R}$ is open, $k \geq 1$, $\forall i < k : a_i : I \rightarrow \mathbb{C}$

Def Homogeneity of LDEs

$$\text{Homogeneous} \iff b = 0$$

$$\text{Inhomogeneous} \iff b \neq 0$$

Remark $D(y) := y^{(k)} + \dots + a_0y$ is a linear operation:

$$D(z_1f_1 + z_2f_2) = z_1D(f_1) + z_2D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}$, f_1, f_2 k -times differentiable

Def Homogeneous Solution Space

$$\mathcal{S}(F) := \{f : I \rightarrow \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1D(f_1) + z_2D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}$, $f_1, f_2 \in \mathcal{S}$

Solutions for complex homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and complex-valued

1. \mathcal{S} is a complex vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and real-valued

1. \mathcal{S} is a real vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def Inhomogeneous Solution Space

$$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), f_0 \text{ is a particular sol.}\}$$

Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous, $b : I \rightarrow \mathbb{C}$

1. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark Applications of Linearity

If f_1 solves F for b_1 , and f_2 for b_2 : $f_1 + f_2$ solves $b_1 + b_2$.

Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

2.2 Linear Solutions: First Order

I \subset \mathbb{R}, \quad a, b : I \rightarrow \mathbb{R}

Form:

y' + ay = b

Approach:

- 1. Hom. Solution f_1 for: y' + ay = 0
Note that S has dim(S) = 1, so f_1 \neq 0 is a Basis for S
- 2. Part. Solution f_0 for y' + ay = b

Solutions: f_0 + z f_1 \quad \text{for } z \in \mathbb{C}

Explicit Homogeneous Solution

A(x) is a primitive of a, f(x_0) = y_0

f_1(x) = z \cdot \exp(-A(x))

f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))

Variation of Constants: Treating z as z(x) yields:

Explicit Inhomogeneous Solution

A(x) is a primitive of a

f_0(x) = \left(\underbrace{\int b(x) \cdot \exp(A(x))}_{z(x)} \right) \cdot \exp(-A(x))

Method Educated Guess
Usually, y has a similar form to b:

b(x)	Guess
a \cdot e^{\alpha x}	b \cdot e^{\alpha x}
a \cdot \sin(\beta x)	c \sin(\beta x) + d \cos(\beta x)
b \cdot \cos(\beta x)	c \sin(\beta x) + d \cos(\beta x)
a e^{\alpha x} \cdot \sin(\beta x)	e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))
b e^{\alpha x} \cdot \cos(\beta x)	e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))
P_n(x) \cdot e^{\alpha x}	R_n(x) \cdot e^{\alpha x}
P_n(x) \cdot e^{\alpha x} \sin(\beta x)	e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))
P_n(x) \cdot e^{\alpha x} \cos(\beta x)	e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))

Remark If \alpha, \beta are roots of P(X) with multiplicity j, multiply guess with a P_j(x).

2.3 Linear Solutions: Constant Coefficients

Form:

y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b

Where a_0, \dots, a_{k-1} \in \mathbb{C} are constants, b(x) is continuous.

2.3.1 Homogeneous Equations

The idea is to find a Basis of S:

Def Characteristic Polynomial P(X) = \prod_{i=1}^k (X - \alpha_i)

Remark The unique roots \alpha_1, \dots, \alpha_l form a Basis:

\text{span}(S) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}

v_1, \dots, v_k are the Multiplicities of \alpha_1, \dots, \alpha_k

Remark If \alpha_j = \beta + \gamma i \in \mathbb{C} is a root, \bar{\alpha}_j = \beta - \gamma i is too. To get a real-valued solution, apply:

e^{\alpha_j x} = e^{\beta x} (\cos(\gamma x) + i \sin(\gamma x))

Explicit Homogeneous Solution

Using \alpha_1, \dots, \alpha_k from P(X) s.t. \alpha_i \neq \alpha_j, z_i \in \mathbb{C} arbitrary

f(x) = \prod_{i=1}^k z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)}(x) = \prod_{i=1}^k z_i \cdot \alpha_i^j e^{\alpha_i x}

Multiple roots: same scheme, using the basis vectors of S

Solutions exist \forall Z = (z_1, \dots, z_k) since that system's \det(M_Z) \neq 0.

2.3.2 Inhomogeneous Equations

Method Undetermined Coefficients: An educated guess.

- 1. b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}
deg(Q) \leq d + v_\alpha, where v_\alpha is \alpha's multiplicity in P(X)
- 2. \left. \begin{aligned} b(x) &= cx^d \cdot \cos(\alpha x) \\ b(x) &= cx^d \cdot \sin(\alpha x) \end{aligned} \right\} f_p = Q_1(x) \cos(\alpha x) + Q_2(x) \sin(\alpha x)
deg(Q_{1,2}) \leq d + v_\alpha, where v_\alpha is \alpha's multiplicity in P(X)

Remark Applying Linearity

If b(x) = \sum_{i=1}^n b_i(x), A solution for b(x) is f(x) = \sum_{i=1}^n f_i(x)
Sometimes called Superposition Principle in this context

2.4 Other Methods

Method Change of Variable

If f(x) is replaced by h(y) = f(g(y)), then h is a sol. too.
Changes like h(t) = f(e^t) may lead to useful properties.

Separation of Variables

Form:

y' = a(y) \cdot b(x)

Solve using:

\int \frac{1}{a(y)} dy = \int b(x) dx + c

Usually \int 1/a(y) dy can be solved directly for \ln |a(y)| + c.

2.5 Method Overview

Method	Use case
Variation of constants	LDE with ord(F) = 1
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. y' = a(y) \cdot b(x)
Change of Variables	e.g. y' = f(ax + by + c)

3 Continuous functions in \mathbb{R}^n

Treating functions $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{C}/\mathbb{R}^m$, $m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ means:
 $x = (x_1, \dots, x_n)$, $f(x) = (f_1(x), \dots, f_m(x))$

3.1 Multivariate functions

Def Linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

In other words: $f(x) = Ax$, $A \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

Def Affine Linear map $f(x) \mapsto Ax + c$

Def Quadratic form $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

In other words: $Q(x) = \sum_{i=0}^n \sum_{j=0}^m (a_{i,j} x_i x_j)$

Def Monomials $M(x) : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto \alpha x_1^{d_1} \dots x_n^{d_n}$

For example: $f(x, y, z) = 16x^2 y z^5$

Def $\deg(M) := e = \sum_{i=1}^n d_i$

For example: $\deg(16x^2 y z^5) = 8$

Def Polynomials $P(x) := \sum_{i=0}^n M_i(x)$

For example: $P(x, y, z) = x^3 + 25x^2 y^6 z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$

For example: $\deg(x^3 + 25x^2 y^6 z + xy) = 9$

Visualisations for some function types:

Def Graph $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$

Only for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Visually, this is a surface in \mathbb{R}^3

Def Vector Plots for $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Points in $(x, y) \in \mathbb{R}^2$ are displayed as vectors $f(x, y)$

3.2 Sequences in \mathbb{R}^n

Def Sequences in \mathbb{R}^n

$(x_k)_{k \geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \dots, x_{k,n})$

Def Convergence in \mathbb{R}^n

$$\lim_{k \rightarrow \infty} (x_k) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

Lem. Equivalent conditions to Convergence

$$(i) \quad \forall i \text{ s.t. } 1 \leq i \leq n : \lim_{k \rightarrow \infty} (x_{k,i}) = y_i$$

$$(ii) \quad \lim_{k \rightarrow \infty} \|x_k - y\| = 0$$

Def Limits at points

$$\lim_{x \neq x_0 \rightarrow x_0} (f(x)) = y \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 : \\ \forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m, \quad x_0 \in X, \quad y \in \mathbb{R}^m$$

The sequence test for Continuity works for point-limits too.

3.3 Continuity in \mathbb{R}^n

Def Continuity in \mathbb{R}^n

f continuous at $x_0 \in X \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

f continuous $\stackrel{\text{def}}{\iff} \forall x \in X : f$ continuous at x

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m$$

Lem. Continuity using Sequences

f continuous at x_0 if and only if:

$$\forall (x_k)_{k \geq 1} : \lim_{k \rightarrow \infty} (x_k) = x_0 \implies \lim_{k \rightarrow \infty} (f(x_k)) = f(x_0)$$

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m$$

Lem. Continuity of Compositions

$f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p$ continuous $\implies g \circ f$ continuous

$$X \subset \mathbb{R}^n, \quad Y \subset \mathbb{R}^m, \quad p \geq 1$$

Lem. Continuity using Coordinate Functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

3.4 Subsets of \mathbb{R}^n

Def Bounded

$X \subset \mathbb{R}^n$ bounded $\stackrel{\text{def}}{\iff} \{\|x\| \mid x \in X\} \subset \mathbb{R}$ bounded.

Example: The open disc $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is bounded.

Def Closed

$X \subset \mathbb{R}^n$ closed $\stackrel{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \lim_{x \rightarrow \infty} (x_k) \in X$

Example: \emptyset, \mathbb{R}^n are closed.

Def Compact if closed and bounded.

Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is compact.

Def Open

$X \subset \mathbb{R}^n$ open $\stackrel{\text{def}}{\iff} \forall x \in X, \exists \delta > 0 :$

$$\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall i \leq n\} \subset X$$

In other words: Changing any coord. x_i by δ keeps x' in X

Example: \emptyset, \mathbb{R}^n are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. Continous functions preserve closed/open

\forall closed/open $Y :$

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous, $Y \subset \mathbb{R}^m$

Lem. The complement of open sets is closed

$$X \subset \mathbb{R}^n \text{ is open} \iff \underbrace{\{x \in \mathbb{R}^n \mid x \notin X\}}_{\text{Complement}} \text{ is closed}$$

Min-Max Theorem

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \rightarrow \mathbb{R}$:

$$\exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), \quad f(x_2) = \inf_{x \in X} f(x)$$

4 Differential Calculus in \mathbb{R}^n

4.1 Partial Derivatives

Partial Derivative

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $1 \leq i \leq n$, $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \rightarrow \mathbb{R}$

$$g(t) := \underbrace{f(x_{0,1}, \dots, x_{0,t-1}, t, x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \rightarrow t}$$

Notation $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

Lem. Properties of Partial Derivatives

Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

- (i) $\partial_{x_i}(f + g) = \partial_{x_i} f + \partial_{x_i} g$
- (ii) $\partial_{x_i}(fg) = \partial_{x_i}(f)g + \partial_{x_i}(g)f$ if $m = 1$
- (iii) $\partial_{x_i}\left(\frac{f}{g}\right) = \frac{\partial_{x_i}(f)g - \partial_{x_i}(g)f}{g^2}$ if $g(x) \neq 0 \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f, g : X \rightarrow \mathbb{R}$, $1 \leq i \leq n$

The Jacobian

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial x_1 f_1(x) & \partial x_2 f_1(x) & \cdots & \partial x_n f_1(x) \\ \partial x_1 f_2(x) & \partial x_2 f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_1 f_m(x) & \partial x_2 f_m(x) & \cdots & \partial x_n f_m(x) \end{bmatrix}$$

Think of f as a vector of f_i , then \mathbf{J}_f is that vector stretched for all x_j

$$\text{Def } \textbf{Gradient } \nabla f(x_0) := \begin{bmatrix} \partial x_1 f(x_0) \\ \vdots \\ \partial x_n f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$

Def Divergence $\text{div}(f)(x_0) := \text{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$, \mathbf{J}_f exists

4.2 The Differential

Partial derivatives don't provide a good approx. of f , unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in \mathbb{R}^n .

Differentiability in \mathbb{R}^n

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$, $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If f is differentiable at $x_0 \in X$ with u s.t.

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{1}{\|x - x_0\|} \left(f(x) - f(x_0) - u(x - x_0) \right) = 0$$