1 Differential Equations

Def **Differential Equation** (DE)
Equation relating unknown f to derivatives $f^{(i)}$ at *same* x.

Def **Ordinary Differential Equation** (ODE)
DE s.t. $f : I \to \mathbb{R}$ is in one variable.

Def **Partial Differential Equation** (PDE)
DE s.t. $f : I^d \to \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def **Order** $\text{ord}(F) := \max_{i \geq 0}\{i \mid f^{(i)} \in F,\ f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \to \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$$\exists f : I \to \mathbb{R}$$
$$\text{s.t. } \forall x \in I : f'(x) = F(x, f(x)) \text{ and } f(x_0) = y_0$$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

1.1 Linear Differential Equations

Def **Linear Differential Equation** (LDE)

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

$I \subset \mathbb{R}$ is open, $k \geq 1$, $\forall i < k : a_i : I \to \mathbb{C}$

Def Homogeneity of LDEs

Homogeneous $\overset{\text{def}}{\iff} b = 0$

Inhomogeneous $\overset{\text{def}}{\iff} b \neq 0$

Remark $D(y) := y^{(k)} + \ldots + a_0 y$ is a linear operation:

$$D(z_1 f_1 + z_2 f_2) = z_1 D(f_1) + z_2 D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}$, f_1, f_2 k-times differentiable:

Def **Homogeneous Solution Space**
$\mathcal{S}(F) := \{f : I \to \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1 D(f_1) + z_2 D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}$, $f_1, f_2 \in \mathcal{S}$

Solutions for complex homogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous and complex-valued
1. $\mathcal{S}$ is a complex vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous and real-valued
1. $\mathcal{S}$ is a real vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def **Inhomogeneous Solution Space**
$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F),\ f_0 \text{ is a particular sol.}\}$
Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous, $b : I \to \mathbb{C}$
1. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark **Applications of Linearity**
If f_1 solves F for b_1, and f_2 for b_2: $f_1 + f_2$ solves $b_1 + b_2$.
Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

1.2 Finding Solutions: First Order

$I \subset \mathbb{R}$, $a, b : I \to \mathbb{R}$

$$y' + ay = b$$

Approach:

1. Hom. Solution: $y' + ay = 0$ using $f_1 = ke^{-A(x)}$
   Note that $\mathcal{S}$ has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for $\mathcal{S}$
2. Part. Solution: $f_0 \in \mathcal{S}_b$ using Variance of Parameters

Solutions: $f_0 + zf_1$ for $z \in \mathbb{C}$

Explicit Solution for 1st Order LDEs

$A(x)$ is a primitive of a, $f(x_0) = y_0$

$$f(x) = z \cdot \exp(-A(x))$$
$$f(x) = y_0 \cdot \exp(A(x_0) - a(x))$$