

# 1 Linear Algebra

Relevant definitions used throughout Analysis II.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

$$\text{Def } \mathbf{Scalar Product} \quad x \cdot y := \sum_{i=0}^n (x_i \cdot y_i)$$

$$\text{Def } \mathbf{Euclidian Norm} \quad \|x\| := \sqrt{\sum_{i=1}^n x_i^2}$$

Used to generalize  $|x|$  in many Analysis I definitions

**Lem.** **Properties of  $\|x\|$**

- (i)  $\|x\| \geq 0$
- (ii)  $\|x\| \iff x = 0$
- (iii)  $\|\alpha x\| = |\alpha| \cdot \|x\|$
- (iv)  $\|x + y\| \leq \|x\| + \|y\|$  (Triangle Inequality)

$$\text{Def } \mathbf{Trace} \quad \text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$$

# 2 Differential Equations

**Def** **Differential Equation (DE)**

Equation relating unknown  $f$  to derivatives  $f^{(i)}$  at *same*  $x$ .

**Def** **Ordinary Differential Equation (ODE)**

DE s.t.  $f : I \rightarrow \mathbb{R}$  is in one variable.

**Def** **Partial Differential Equation (PDE)**

DE s.t.  $f : I^d \rightarrow \mathbb{R}$  is in multiple variables.

**Notation**  $f^{(i)}$  or  $y^{(i)}$  instead of  $f^{(i)}(x)$  for brevity.

**Def** **Order**  $\text{ord}(F) := \max_{i \geq 0} \{i \mid f^{(i)} \in F, f^{(i)} \neq 0\}$

**Remark** Any  $F$  s.t.  $\text{ord}(F) \geq 2$  can be reduced to  $\text{ord}(F') = 1$ , but using functions of higher dimensions.

## Solutions to ODEs

$\forall F : \mathbb{R}^2 \rightarrow \mathbb{R}$  s.t.  $F$  is cont. diff. and  $x_0, y_0 \in \mathbb{R}$ :

$$\exists f : I \rightarrow \mathbb{R}$$

$$\text{s.t. } \forall x \in I : f'(x) = F(x, f(x)) \text{ and } f(x_0) = y_0$$

s.t.  $I$  is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

## 2.1 Linear Differential Equations

**Def** **Linear Differential Equation (LDE)**

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

$I \subset \mathbb{R}$  is open,  $k \geq 1$ ,  $\forall i < k : a_i : I \rightarrow \mathbb{C}$

**Def** **Homogeneity of LDEs**

**Homogeneous**  $\iff b = 0$

**Inhomogeneous**  $\iff b \neq 0$

**Remark**  $D(y) := y^{(k)} + \dots + a_0y$  is a linear operation:

$$D(z_1f_1 + z_2f_2) = z_1D(f_1) + z_2D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$  *k*-times differentiable

**Def** **Homogeneous Solution Space**

$\mathcal{S}(F) := \{f : I \rightarrow \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

**Remark**  $\mathcal{S}(F)$  is the Nullspace of a lin. map:  $f$  to  $D(f)$ :

$$D(f) = z_1D(f_1) + z_2D(f_2) = 0$$

$$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$$

## Solutions for complex homogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous and complex-valued

1.  $\mathcal{S}$  is a complex vector space,  $\dim(\mathcal{S}) = k$
2.  $\mathcal{S}$  is a subspace of  $\{f \mid f : I \rightarrow \mathbb{C}\}$
3.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$  a unique sol. exists

## Solutions for real homogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous and real-valued

1.  $\mathcal{S}$  is a real vector space,  $\dim(\mathcal{S}) = k$
2.  $\mathcal{S}$  is a subspace of  $\{f \mid f : I \rightarrow \mathbb{R}\}$
3.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{R}^k$  a unique sol. exists

**Def** **Inhomogeneous Solution Space**

$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), f_0 \text{ is a particular sol.}\}$

Note: This is only a vector space if  $b = 0$ , where  $\mathcal{S}_b = \mathcal{S}$ .

## Solutions for real inhomogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous,  $b : I \rightarrow \mathbb{C}$

1.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$  a unique sol. exists
2. If  $b, a_i$  are real-valued, a real-valued sol. exists.

**Remark** **Applications of Linearity**

If  $f_1$  solves  $F$  for  $b_1$ , and  $f_2$  for  $b_2$ :  $f_1 + f_2$  solves  $b_1 + b_2$ .  
Follows from:  $D(f_1) + D(f_2) = b_1 + b_2$ .

### 3 Solutions to Differential Equations

#### 3.1 Linear Solutions: First Order

**Form:**  $y' + ay = b \quad I \subset \mathbb{R}, \quad a, b : I \rightarrow \mathbb{R}$

**Approach:**

1. Hom. Solution  $f_1$  for:  $y' + ay = 0$

Note that  $\mathcal{S}$  has  $\dim(\mathcal{S}) = 1$ , so  $f_1 \neq 0$  is a Basis for  $\mathcal{S}$

2. Part. Solution  $f_0$  for  $y' + ay = b$

**Solutions:**  $f_0 + zf_1 \quad \text{for } z \in \mathbb{C}$

##### Explicit Homogeneous Solution

$A(x)$  is a primitive of  $a$ ,  $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$

$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

**Method Variation of Constants:** Treating  $z$  as  $z(x)$  yields:

##### Explicit Inhomogeneous Solution

$A(x)$  is a primitive of  $a$

$$f_0(x) = \underbrace{\left( \int b(x) \cdot \exp(A(x)) \right)}_{z(x)} \cdot \exp(-A(x))$$

##### Method Educated Guess

Usually,  $y$  has a similar form to  $b$ :

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \cdot \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cdot \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$a e^{\alpha x} \cdot \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$b e^{\alpha x} \cdot \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

**Remark** If  $\alpha, \beta$  are roots of  $P(X)$  with multiplicity  $j$ , multiply guess with a  $P_j(x)$ .

#### 3.2 Linear Solutions: Constant Coefficients

**Form:**

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Where  $a_0, \dots, a_{k-1} \in \mathbb{C}$  are constants,  $b(x)$  is continuous.

##### 3.2.1 Homogeneous Equations

The idea is to find a Basis of  $\mathcal{S}$ :

**Def** **Characteristic Polynomial**  $P(X) = \prod_{i=1}^k (X - \alpha_i)$

**Remark** The unique roots  $\alpha_1, \dots, \alpha_l$  form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}$$

$v_1, \dots, v_k$  are the Multiplicities of  $\alpha_1, \dots, \alpha_k$

**Remark** If  $\alpha_j = \beta + \gamma i \in \mathbb{C}$  is a root,  $\bar{\alpha}_j = \beta - \gamma i$  is too. To get a real-valued solution, apply:

$$e^{\alpha_j x} = e^{\beta x} (\cos(\gamma x) + i \sin(\gamma x))$$

##### Explicit Homogeneous Solution

Using  $\alpha_1, \dots, \alpha_k$  from  $P(X)$  s.t.  $\alpha_i \neq \alpha_j$ ,  $z_i \in \mathbb{C}$  arbitrary

$$f(x) = \prod_{i=1}^k z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)(x)} = \prod_{i=1}^k z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of  $\mathcal{S}$

Solutions exist  $\forall Z = (z_1, \dots, z_k)$  since that system's  $\det(M_Z) \neq 0$ .

#### 3.2.2 Inhomogeneous Equations

**Method Undetermined Coefficients:** An educated guess.

1.  $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$   
 $\deg(Q) \leq d + v_\alpha$ , where  $v_\alpha$  is  $\alpha$ 's multiplicity in  $P(X)$

2.  $b(x) = cx^d \cdot \cos(\alpha x)$   
 $b(x) = cx^d \cdot \sin(\alpha x)$   
 $\deg(Q_{1,2}) \leq d + v_\alpha$ , where  $v_\alpha$  is  $\alpha$ 's multiplicity in  $P(X)$

**Remark Applying Linearity**

If  $b(x) = \sum_{i=1}^n b_i(x)$ , A solution for  $b(x)$  is  $f(x) = \sum_{i=1}^n f_i(x)$   
Sometimes called *Superposition Principle* in this context

#### 3.3 Other Methods

**Method Change of Variable**

If  $f(x)$  is replaced by  $h(y) = f(g(y))$ , then  $h$  is a sol. too.  
Changes like  $h(t) = f(e^t)$  may lead to useful properties.

##### Separation of Variables

Form:

$$y' = a(y) \cdot b(x)$$

Solve using:

$$\int \frac{1}{a(y)} dy = \int b(x) dx + c$$

Usually  $\int 1/a(y) dy$  can be solved directly for  $\ln|a(y)| + c$ .

#### 3.4 Method Overview

Method	Use case
Variation of constants	LDE with $\text{ord}(F) = 1$
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. $y' = a(y) \cdot b(x)$
Change of Variables	e.g. $y' = f(ax + by + c)$

## 4 Continuous functions in $\mathbb{R}^n$

Treating functions  $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{C}/\mathbb{R}^m$ ,  $m, n \geq 1$

**Notation**  $f(x)$  for  $f : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  means:  
 $x = (x_1, \dots, x_n)$ ,  $f(x) = (f_1(x), \dots, f_m(x))$

### 4.1 Multivariate functions

**Def** **Linear map**  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

In other words:  $f(x) = \mathbf{A}x$ ,  $\mathbf{A} \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

**Def** **Affine Linear map**  $f(x) \mapsto \mathbf{A}x + c$

**Def** **Quadratic form**  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

In other words:  $Q(x) = \sum_{i=0}^n \sum_{j=0}^m (a_{i,j} x_i x_j)$

**Def** **Monomials**  $M(x) : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$

For example:  $f(x, y, z) = 16x^2yz^5$

**Def**  $\deg(M) := e = \sum_{i=1}^n d_i$

For example:  $\deg(16x^2yz^5) = 8$

**Def** **Polynomials**  $P(x) := \sum_{i=0}^n M_i(x)$

For example:  $P(x, y, z) = x^3 + 25x^2y^6z + xy$

Polynomials are continuous.

**Def**  $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$

For example:  $\deg(x^3 + 25x^2y^6z + xy) = 9$

Visualisations for some function types:

**Def** **Graph**  $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$

Only for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Visually, this is a surface in  $\mathbb{R}^3$

**Def** **Vector Plots** for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Points in  $(x, y) \in \mathbb{R}^2$  are displayed as vectors  $f(x, y)$

### 4.2 Sequences in $\mathbb{R}^n$

**Def** **Sequences in  $\mathbb{R}^n$**

$(x_k)_{k \geq 1}$  s.t.  $x_k \in \mathbb{R}^n$  where  $x_k = (x_{k,1}, \dots, x_{k,n})$

**Def** **Convergence in  $\mathbb{R}^n$**

$$\lim_{k \rightarrow \infty} (x_k) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

**Lem.** **Equivalent conditions to Convergence**

- (i)  $\forall i \text{ s.t. } 1 \leq i \leq n : \lim_{k \rightarrow \infty} (x_{k,i}) = y_i$
- (ii)  $\lim_{k \rightarrow \infty} \|x_k - y\| = 0$

**Def** **Limits at points**

$$\lim_{x \neq x_0 \rightarrow x_0} (f(x)) = y \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 : \forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$X \subset \mathbb{R}^n, f : X \rightarrow \mathbb{R}^m, x_0 \in X, y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

### 4.3 Continuity in $\mathbb{R}^n$

**Def** **Continuity in  $\mathbb{R}^n$**

$f$  continuous at  $x_0 \in X \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

$f$  continuous  $\stackrel{\text{def}}{\iff} \forall x \in X : f$  continuous at  $x$

$X \subset \mathbb{R}^n, f : X \rightarrow \mathbb{R}^m$

**Lem.** **Continuity using Sequences**

$f$  continuous at  $x_0$  if and only if:

$$\forall (x_k)_{k \geq 1} : \lim_{k \rightarrow \infty} (x_k) = x_0 \implies \lim_{k \rightarrow \infty} (f(x_k)) = f(x_0)$$

$X \subset \mathbb{R}^n, f : X \rightarrow \mathbb{R}^m$

**Lem.** **Continuity of Compositions**

$f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p$  continuous  $\implies g \circ f$  continuous

$X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m, p \geq 1$

**Lem.** **Continuity using Coordinate Functions**

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  continuous  $\iff \forall i \leq m : f_i$  continuous

### 4.4 Subsets of $\mathbb{R}^n$

**Def** **Bounded**

$X \subset \mathbb{R}^n$  bounded  $\stackrel{\text{def}}{\iff} \{\|x\| \mid x \in X\} \subset \mathbb{R}$  bounded.

Example: The open disc  $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$  is bounded.

**Def** **Closed**

$X \subset \mathbb{R}^n$  closed  $\stackrel{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \lim_{k \rightarrow \infty} (x_k) \in X$

Example:  $\emptyset, \mathbb{R}^n$  are closed.

**Def** **Compact** if closed and bounded.

Example: The closed Disc  $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$  is compact.

**Def** **Open**

$X \subset \mathbb{R}^n$  open  $\stackrel{\text{def}}{\iff} \forall x \in X, \exists \delta > 0 :$

$$\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall i \leq n\} \subset X$$

In other words: Changing any coord.  $x_i$  by  $\delta$  keeps  $x'$  in  $X$

Example:  $\emptyset, \mathbb{R}^n$  are open (and closed)

**Lem.** The Cartesian Product preserves bounded/closed.

**Lem.** **Continuous functions preserve closed/open**

$\forall$  closed/open  $Y :$

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous,  $Y \subset \mathbb{R}^m$

**Lem.** **The complement of open sets is closed**

$X \subset \mathbb{R}^n$  is open  $\iff \underbrace{\{x \in \mathbb{R}^n \mid x \notin X\}}_{\text{Complement}}$  is closed

### Min-Max Theorem

For compact, non-empty  $X \subset \mathbb{R}^n$ , continuous  $f : X \rightarrow \mathbb{R}$ :

$$\exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), f(x_2) = \inf_{x \in X} f(x)$$

# 5 Differential Calculus in $\mathbb{R}^n$

## 5.1 Partial Derivatives

### Partial Derivative

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ ,  $1 \leq i \leq n$ ,  $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for  $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \rightarrow \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,1}, \dots, x_{0,t-1}, t, x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \rightarrow t}$$

Notation  $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

### Lem. Properties of Partial Derivatives

Assuming  $\partial_{x_i} f$  and  $\partial_{x_i} g$  exist:

$$(i) \quad \partial_{x_i}(f+g) = \partial_{x_i} f + \partial_{x_i} g$$

$$(ii) \quad \partial_{x_i}(fg) = \partial_{x_i}(f)g + \partial_{x_i}(g)f \quad \text{if } m=1$$

$$(iii) \quad \partial_{x_i}\left(\frac{f}{g}\right) = \frac{\partial_{x_i}(f)g - \partial_{x_i}(g)f}{g^2} \quad \text{if } g(x) \neq 0 \quad \forall x \in X$$

$X \subset \mathbb{R}^n$  open,  $f, g : X \rightarrow \mathbb{R}^n$ ,  $1 \leq i \leq n$

### The Jacobian

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$  with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \cdots & \partial_{x_n} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(x) & \partial_{x_2} f_m(x) & \cdots & \partial_{x_n} f_m(x) \end{bmatrix}$$

Think of  $f$  as a vector of  $f_i$ , then  $\mathbf{J}_f$  is that vector stretched for all  $x_j$

$$\text{Def Gradient } \nabla f(x_0) := \begin{bmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{bmatrix} = \mathbf{J}_f(x_0)^\top$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ , i.e. must map to 1 dimension

**Remark**  $\nabla f$  points in the direction of greatest increase.

This generalizes that in  $\mathbb{R}$ ,  $\text{sgn}(f)$  shows if  $f$  increases/decreases

**Def Divergence**  $\text{div}(f)(x_0) := \text{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$ ,  $\mathbf{J}_f$  exists

## 5.2 The Differential

Partial derivatives don't provide a good approx. of  $f$ , unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in  $\mathbb{R}^n$ .

### Differentiability in $\mathbb{R}^n$ & the Differential

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$ ,  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear map

$$df(x_0) := u$$

If  $f$  is differentiable at  $x_0 \in X$  with  $u$  s.t.

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{1}{\|x - x_0\|} \left( f(x) - f(x_0) - u(x - x_0) \right) = 0$$

Similarly,  $f$  is differentiable if this holds for all  $x \in X$

### Lem. Properties of Differentiable Functions

(i) Continuous on  $X$

(ii)  $\forall i \leq m, j \leq n : \partial_{x_j} f_i$  exists

(iii)  $m = 1 : \partial_{x_i} f(x_0) = a_i$

for:  $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  differentiable on  $X$

### Lem. Preservation of Differentiability

(i)  $f+g$  is differentiable:  $d(f+g) = df + dg$

(ii)  $fg$  is differentiable, if  $m=1$

(iii)  $\frac{f}{g}$  is differentiable, if  $m=1$ ,  $g(x) \neq 0 \quad \forall x \in X$

$X \subset \mathbb{R}^n$  open,  $f, g : X \rightarrow \mathbb{R}^m$  differentiable on  $X$

### Lem. Cont. Partial Derivatives imply Differentiability

if all  $\partial_{x_j} f_i$  exist and are continuous:

$f$  differentiable on  $X$ ,  $df(x_0) = \mathbf{J}_f(x_0)$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$

**Lem. Chain Rule**  $g \circ f$  is differentiable on  $X$

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ \mathbf{J}_{g \circ f}(x_0) &= \mathbf{J}_g(f(x_0)) \cdot \mathbf{J}_f(x_0) \end{aligned}$$

$X \subset \mathbb{R}^n$  open,  $Y \subset \mathbb{R}^m$  open,  $f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p$ ,  $f, g$  diff.-able

### Def Tangent Space

$$T_f(x_0) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0) \right\}$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  diff.-able,  $x_0 \in X$ ,  $u = df(x_0)$

### Def Directional Derivative

$$D_v f(x_0) = \lim_{t \neq 0 \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$ ,  $v \neq 0 \in \mathbb{R}^n$ ,  $x_0 \in X$

### Lem. Directional Derivatives for Diff.-able Functions

$$D_v f(x_0) = df(x_0)(v) = \mathbf{J}_f(x_0) \cdot v$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  diff.-able,  $v \neq 0 \in \mathbb{R}^n$ ,  $x_0 \in X$

**Remark**  $D_v f$  is linear w.r.t  $v$ , so:  $D_{v_1+v_2} f = D_{v_1} f + D_{v_2} f$

**Remark**  $D_v f(x_0) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$

In the case  $f : X \rightarrow \mathbb{R}$ , where  $\theta$  is the angle between  $v$  and  $\nabla f(x_0)$

### 5.3 Higher Derivatives

#### Def Differentiability Classes

$$\begin{aligned} f \in C^1(X; \mathbb{R}^m) &\iff f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \text{ exist} \\ f \in C^k(X; \mathbb{R}^m) &\iff f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \in C^{k-1} \\ f \in C^\infty(X; \mathbb{R}^m) &\iff f \in C^k(X; \mathbb{R}^m) \ \forall k \geq 1 \end{aligned}$$

$$X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}^m$$

**Lem.** Polynomials, Trig. functions and  $\exp$  are in  $C^\infty$

#### Lem. Operations preserve Differentiability Classes

$$\begin{aligned} (i) \quad f + g &\in C^k \\ (ii) \quad fg &\in C^k \quad \text{if } m = 1 \\ (iii) \quad \frac{f}{g} &\in C^k \quad \text{if } m = 1, g(x) \neq 0 \ \forall x \in X \\ f, g \in C^k \end{aligned}$$

#### Lem. Composition preserves Differentiability Classes

$$g \circ f \in C^k$$

$$f \in C^k, \quad f(X) \subset Y, \quad Y \subset \mathbb{R}^m \text{ open, } g : Y \rightarrow \mathbb{R}^p, \quad g \in C^k$$

#### Partial Derivatives commute in $C^k$

$$k \geq 2, \quad X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}^m, \quad f \in C^k$$

$$\forall x, y : \quad \partial_{x,y} f = \partial_{y,x} f$$

This generalizes for  $\partial_{x_1, \dots, x_n} f$ .

#### Remark Linearity of Partial Derivatives

$$\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$$

Assuming both  $\partial_x f_{1,2}$  exist.

#### Def Laplace Operator

$$\Delta f := \operatorname{div}(\nabla f(x)) = \sum_{i=0}^n \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right) = \sum_{i=0}^n \frac{\partial^2 f}{\partial x_i^2}$$

#### The Hessian

$$X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$$

$$\mathbf{H}_f(x) := \begin{bmatrix} \partial_{1,1} f(x_0) & \partial_{2,1} f(x_0) & \cdots & \partial_{n,1} f(x_0) \\ \partial_{1,2} f(x_0) & \partial_{2,2} f(x_0) & \cdots & \partial_{n,2} f(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1,n} f(x_0) & \partial_{2,n} f(x_0) & \cdots & \partial_{n,n} f(x_0) \end{bmatrix}$$

$$\text{Where } (\mathbf{H}_f(x))_{i,j} = \partial_{x_i, x_j} f(x)$$

Note that  $f : X \rightarrow \mathbb{R}$ , i.e.  $\mathbf{H}_f$  only exists for 1-dimensionally valued  $f$

**Notation**  $\mathbf{H}_f(x) = \operatorname{Hess}_f(x) = \nabla^2 f(x)$

**Remark**  $\mathbf{H}_f(x_0)$  is symmetric:  $(\mathbf{H}_f(x_0))_{i,j} = (\mathbf{H}_f(x_0))_{j,i}$

### 5.4 Change of Variable