Analysis II

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October 19, 2025

TITLE PAGE COMING SOON

"Multiply it by ai"

- Özlem Imamoglu, 2025

HS2025, ETHZ

Cheat-Sheet based on Lecture notes and Script

`https://metaphor.ethz.ch/x/2025/hs/401-0213-16L/sc/script-analysis-II.pdf`

Contents

1 Introduction

This Cheat-Sheet does not serve as a replacement for solving exercises and getting familiar with the content. There is no guarantee that the content is 100% accurate, so use at your own risk. If you discover any errors, please open an issue or fix the issue yourself and then open a Pull Request here:

`https://github.com/janishutz/eth-summaries`

This Cheat-Sheet was designed with the HS2025 page limit of 10 A4 pages in mind. Thus, the whole Cheat-Sheet can be printed full-sized, if you exclude the title page, contents and this page. You could also print it as two A5 pages per A4 page and also print the Analysis I summary in the same manner, allowing you to bring both to the exam.

And yes, she did really miss an opportunity there with the quote. . . But she was also sick, so it's not as unexpected

2 Differential Equations

2.1 Introduction

Ex 2.1.1: $f'(x) = f(x)$ has only solution $f(x) = ae^x$ for any $a \in \mathbb{R}$; $f' - a = 0$ has only solution $f(x) = \int_{x_0}^{x} a(t)\, \mathrm{d}t$

T 2.1.2: Let $F: \mathbb{R}^2 \to \mathbb{R}$ be a differential function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. The Ordinary Differential Equation (ODE) $y' = F(x, y)$ has a unique solution f defined on a "largest" interval I that contains x_0 such that $y_0 = f(x_0)$

2.2 Linear Differential Equations

An ODE is considered linear if and only if the ys are only scaled and not part of powers.

D 2.2.1: *(Linear differential equation of order k)* (order = highest derivative) $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$, with a_i and b functions in x. If $b(x) = 0\ \forall x$, ***homogeneous***, else ***inhomogeneous***

T 2.2.2: For open $I \subseteq \mathbb{R}$ and $k \geq 1$, for lin. ODE over I with continuous a_i we have:
1. Set $\mathcal{S}$ of $k\times$ diff. sol. $f: I \to \mathbb{C}(\mathbb{R})$ of the eq. is a complex (real) subspace of complex (real)-valued func. over I
2. $\dim(\mathcal{S}) = k\ \forall x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, exists unique $f \in \mathcal{S}$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)}(x_0) = y_{k-1}$. If a_i real-valued, same applies, but $\mathbb{C}$ replaced by $\mathbb{R}$.
3. Let b continuous on I. Exists solution f_0 to inhom. lin. ODE and $\mathcal{S}_b$ is set of funct. $f + f_0$ where $f \in \mathcal{S}$

The solution space $\mathcal{S}$ is spanned by k functions, which thus form a basis of $\mathcal{S}$. If inhomogeneous, $\mathcal{S}$ not vector space.

Finding solutions (in general)
(1) Find basis $\{f_1, \ldots, f_k\}$ for $\mathcal{S}_0$ for homogeneous equation (set $b(x) = 0$) (i.e. find homogeneous part, solve it)
(2) If inhomogeneous, find f_p that solves the equation. The set of solutions is then $\mathcal{S}_b = \{f_h + f_p \mid f_h \in \mathcal{S}_0\}$.
(3) If there are initial conditions, find equations $\in \mathcal{S}_b$ which fulfill conditions using SLE (as always)

2.3 Linear differential equations of first order

P 2.3.1: Solution of $y' + ay = 0$ is of form $f(x) = ze^{-A(x)}$ with A anti-derivative of a

Imhomogeneous equation
1. Plug all values into $y_p = \int b(x)e^{A(x)}$ ($A(x)$ in the exponent instead of $-A(x)$ as in the homogeneous solution)
2. Solve and the final $y(x) = y_h + y_p$. For initial value problem, determine coefficient z

2.4 Linear differential equations with constant coefficients

The coefficients a_i are constant functions of form $a_i(x) = k$ with k constant, where $b(x)$ can be any function.

Homogeneous Equation
1. Find *characteristic polynomial* (of form $\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_1\lambda + a_0$ for order k lin. ODE with coefficients $a_i \in \mathbb{R}$).
2. Find the roots of polynomial. The solution space is given by $\{z_j \cdot x^{v_j - 1}e^{\gamma_i x} \mid v_j \in \mathbb{N}, \gamma_i \in \mathbb{R}\}$ where v_j is the multiplicity of the root γ_i. For $\gamma_i = \alpha + \beta i \in \mathbb{C}$, we have $z_1 \cdot e^{\alpha x}\cos(\beta x)$, $z_2 \cdot e^{\alpha x}\sin(\beta x)$, representing the two complex conjugated solutions.

Inhomogeneous Equation
1. **(Case 1)** $b(x) = cx^d e^{\alpha x}$, with special cases x^d and $e^{\alpha x}$: $f_p = Q(x)e^{\alpha x}$ with Q a polynomial with $\deg(Q) \leq j + d$, where j is multiplicity of root α (if $P(\alpha) \neq 0$, then $j = 0$) of characteristic polynomial
2. **(Case 2)** $b(x) = cx^d\cos(\alpha x)$, or $b(x) = cx^d\sin(\alpha x)$: $f_p = Q_1(x) \cdot \cos(\alpha x) + Q_2(x9 \cdot \sin(\alpha x))$, where $Q_i(x)$ a polynomial with $\deg(Q_i) \leq d + j$, where j is the multiplicity of root αi (if $P(\alpha i) \neq 0$, then $j = 0$) of characteristic polynomial

Other methods
- ***Change of variable*** Apply substitution method here, substituting for example for $y' = f(ax + by + c)$ $u = ax + by$ to make the integral simpler. Mostly intuition-based (as is the case with integration by substitution)
- ***Separation of variables*** For equations of form $y' = a(y) \cdot b(x)$ (NOTE: Not linear), we transform into $\frac{y'}{a(y)} = b(x)$ and then integrate by substituting $y'(x)dx = dy$, changing the variable of integration. Solution: $A(y) = B(x) + c$, with $A = \int \frac{1}{a}$ and $B(x) = \int b(x)$. To get final solution, solve for the above equation for y.

3 Differential Calculus in Vector Space

3.2 Continuity

D 3.2.1: *(Convergence in $\mathbb{R}^n$)* Let $(x_k)_{k \in \mathbb{N}}$ where $x_k \in \mathbb{R}^n$ with $x_k = (x_{k,1}, \ldots, x_{k,n})$ and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. (x_k) converges to y as $k \to +\infty$ if $\forall \varepsilon > 0\ \exists N \geq 1$ s.t. $\forall n \geq N$ we have $||x_k - y|| < \varepsilon$ **L 3.2.2:** (x_k) converges to y as $k \to +\infty$ iff one of following equiv. statements holds: **(1)** $\forall 1 \leq i \leq n$, the sequence $(x_{k,i})$ with $x_{k,i} \in \mathbb{R}$ converges to y_i **(2)** $(||x_k - y||)$ converges to 0 as $k \to +\infty$ **D 3.2.3:** *(Continuity)* Let $X \subseteq \mathbb{R}^n$ and $f: X \to \mathbb{R}^m$. **(1)** Let $x_0 \in X$. f continuous in $\mathbb{R}^n$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $x \in X$ satisfies $||x - x_0|| < \delta$, then $||f(x) - f(x_0)|| < \varepsilon$ **(2)** f continuous *on* X if continuous at $x_0\ \forall x_0 \in X$ **P 3.2.4:** Let X and f as prev. Let $x_0 \in X$. f continuous at x_0 iff $\forall (x_k)_{k \geq 1}$ in X s.t. $x_k \to x_0$ as $k \to +\infty$, $(f(x_k))_{k \geq 1}$ in $\mathbb{R}^m$ converges to $f(x)$ **D 3.2.5:** *(Limit)* Let X, f and x_0 as prev. and $y \in \mathbb{R}^m$. f *has limit* y as $x \to x_0$ with $x \neq x_0$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \neq x_0 \in X, ||x - x_0|| < \delta$ we have $||f(x) - y|| < \varepsilon$. We write $\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y$ **R 3.2.6:** Also possible without ass. that $x_0 \in X$

P 3.2.7: Let X, f, x_0 and y as prev. We have $\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y$ iff $\forall (x_k)$ in X s.t. $x_k \to x$ as $k \to +\infty$ and $x_k \neq x_0$ $(f(x_k))$ in $\mathbb{R}^m$ converges to y **P 3.2.9:** Let $X \subseteq \mathbb{R}^n$, $y \subseteq \mathbb{R}^m$, $p \in \mathbb{N}$ and let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be cont. Then $g \circ f$ is continuous

Ex 3.2.10: *(1)* $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ continuous $\Rightarrow f = (f_1, f_2) : \mathbb{R}^n \to \mathbb{R}^{m_1 + m_2}$ is continuous (Cartesian product) *(2)* Any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. In particular, the identity map is continuous *(3)* If $f_1, \ldots, f_n$ continuous, then $f(x_1, \ldots, x_n) = f_1(x_1) \cdot \ldots \cdot f_n(x_n)$ is continuous *(4)* Polynomials in $x_1, \ldots, x_n$ are continuous *(5)* $f_1 f_2$ is continuous if f_1 and f_2 are continuous and if $f_2(x) \neq 0$ $\forall x \in X$, then $f_1 \div f_2$ is continuous. (see Theorem 2.1.8 in Analysis I) *(6)* If both f and g have limits, then $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$ and analogous for $\times$ *(7)* If $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, then $g(x) = f(x, y_0)$ for $y_0 \in \mathbb{R}$ is continuous. The converse is not true

D 3.2.11: *(1)* $X \subseteq \mathbb{R}^n$ is **bounded** if the set of $||x||$ for $x \in X$ is bounded in $\mathbb{R}$ *(2)* $X \subseteq \mathbb{R}^n$ is **closed** if $\forall (x_k)$ in X that converge in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$ *(3)* $X \subseteq \mathbb{R}^n$ is **compact** if it is bounded and closed

Ex 3.2.12: *(1)* $\emptyset$ and $\mathbb{R}^n$ are closed. *(2)* The *open* disc $D = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$ for $r > 0$ and $x_0 \in \mathbb{R}^n$ is bounded and not closed. *(3)* The *closed* disc $\Delta = \{x \in \mathbb{R}^n : ||x - x_0|| \leq r\}$ is bounded and closed. In particular, a closed interval is a closed set. An interval is compact if it is bounded *(4)* If $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^m$ are bounded (also closed or compact), then so is $X_1 \times X_2 \subseteq \mathbb{R}^{n+m}$

P 3.2.13: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed $Y \subseteq \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subseteq \mathbb{R}^n$ is closed

Ex 3.2.14: The **zero set** $Z = \{x \in \mathbb{R}^n : f(x) = 0\}$ is closed in $\mathbb{R}^n$ because $\{0\} \subseteq \mathbb{R}$ is closed. More generally: for any $r \geq 0$, $\{x \in \mathbb{R}^n : |f(x)| \leq r\}$ is $f^{-1}([-r, r])$ and is closed, since $[-r, r]$ is closed. Furthermore: $\{x \in \mathbb{R}^3 : ||x - x_0|| = r\}$ is closed

T 3.2.15: Let $(X \neq \emptyset) \subseteq \mathbb{R}^n$ compact and $f : X \to \mathbb{R}$ continuous. Then f bounded, has max and min, i.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup_{x \in X} f(x)$ and $f(x_-) = \inf_{x \in X} f(x)$

3.3   Partial derivatives