Analysis II

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TITLE PAGE COMING SOON

"*Multiply it by ai*"

- Özlem Imamoglu, 2025

HS2025, ETHZ

Cheat-Sheet based on Lecture notes and Script

https://metaphor.ethz.ch/x/2025/hs/401-0213-16L/sc/script-analysis-II.pdf

Contents

1 Introduction

This Cheat-Sheet does not serve as a replacement for solving exercises and getting familiar with the content. There is no guarantee that the content is 100% accurate, so use at your own risk. If you discover any errors, please open an issue or fix the issue yourself and then open a Pull Request here:

`https://github.com/janishutz/eth-summaries`

This Cheat-Sheet was designed with the HS2025 page limit of 10 A4 pages in mind. Thus, the whole Cheat-Sheet can be printed full-sized, if you exclude the title page, contents and this page. You could also print it as two A5 pages per A4 page and also print the Analysis I summary in the same manner, allowing you to bring both to the exam.

And yes, she did really miss an opportunity there with the quote... But she was also sick, so it's not as unexpected

2 Differential Equations

2.1 Introduction

Ex 2.1.1: $f'(x) = f(x)$ has only solution $f(x) = ae^x$ for any $a \in \mathbb{R}$; $f' - a = 0$ has only solution $f(x) = \int_{x_0}^{x} a(t)\, dt$

T 2.1.2: Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a differential function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. The Ordinary Differential Equation (ODE) $y' = F(x, y)$ has a unique solution f defined on a "largest" interval I that contains x_0 such that $y_0 = f(x_0)$

2.2 Linear Differential Equations

An ODE is considered linear if and only if the ys are only scaled and not part of powers.
D 2.2.1: *(Linear differential equation of order k)* (order = highest derivative) $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$, with a_i and b functions in x. If $b(x) = 0 \ \forall x$, ***homogeneous***, else ***inhomogeneous***
T 2.2.2: For open $I \subseteq \mathbb{R}$ and $k \geq 1$, for lin. ODE over I with continuous a_i we have:

1. Set $\mathcal{S}$ of $k\times$ diff. sol. $f : I \to \mathbb{C}(\mathbb{R})$ of the eq. is a complex (real) subspace of complex (real)-valued func. over I
2. $\dim(\mathcal{S}) = k \ \forall x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, exists unique $f \in \mathcal{S}$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)}(x_0) = y_{k-1}$. If a_i real-valued, same applies, but $\mathbb{C}$ replaced by $\mathbb{R}$.
3. Let b continuous on I. Exists solution f_0 to inhom. lin. ODE and $\mathcal{S}_b$ is set of funct. $f + f_0$ where $f \in \mathcal{S}$

The solution space $\mathcal{S}$ is spanned by k functions, which thus form a basis of $\mathcal{S}$. If inhomogeneous, $\mathcal{S}$ not vector space.

Finding solutions (in general)

(1) Find basis $\{f_1, \ldots, f_k\}$ for $\mathcal{S}_0$ for homogeneous equation (set $b(x) = 0$) (i.e. find homogeneous part, solve it)
(2) If inhomogeneous, find f_p that solves the equation. The set of solutions is then $\mathcal{S}_b = \{f_h + f_p \mid f_h \in \mathcal{S}_0\}$.
(3) If there are initial conditions, find equations $\in \mathcal{S}_b$ which fulfill conditions using SLE (as always)

2.3 Linear differential equations of first order

P 2.3.1: Solution of $y' + ay = 0$ is of form $f(x) = ze^{-A(x)}$ with A anti-derivative of a

Imhomogeneous equation

1. Plug all values into $y_p = \int b(x)e^{A(x)}$ ($A(x)$ in the exponent instead of $-A(x)$ as in the homogeneous solution)
2. Solve and the final $y(x) = y_h + y_p$. For initial value problem, determine coefficient z

2.4 Linear differential equations with constant coefficients

The coefficients a_i are constant functions of form $a_i(x) = k$ with k constant, where $b(x)$ can be any function.

Homogeneous Equation

1. Find ***characteristic polynomial*** (of form $\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_1\lambda + a_0$ for order k lin. ODE with coefficients $a_i \in \mathbb{R}$).
2. Find the roots of polynomial. The solution space is given by $\{z_j \cdot x^{v_j - 1}e^{\gamma_i x} \mid v_j \in \mathbb{N}, \gamma_i \in \mathbb{R}\}$ where v_j is the multiplicity of the root γ_i. For $\gamma_i = \alpha + \beta i \in \mathbb{C}$, we have $z_1 \cdot e^{\alpha x}\cos(\beta x)$, $z_2 \cdot e^{\alpha x}\sin(\beta x)$, representing the two complex conjugated solutions.

Inhomogeneous Equation

1. ***(Case 1)*** $b(x) = cx^d e^{\alpha x}$, with special cases x^d and $e^{\alpha x}$: $f_p = Q(x)e^{\alpha x}$ with Q a polynomial with $\deg(Q) \leq j + d$, where j is multiplicity of root α (if $P(\alpha) \neq 0$, then $j = 0$) of characteristic polynomial
2. ***(Case 2)*** $b(x) = cx^d\cos(\alpha x)$, or $b(x) = cx^d\sin(\alpha x)$: $f_p = Q_1(x) \cdot \cos(\alpha x) + Q_2(x9 \cdot \sin(\alpha x))$, where $Q_i(x)$ a polynomial with $\deg(Q_i) \leq d + j$, where j is the multiplicity of root αi (if $P(\alpha i) \neq 0$, then $j = 0$) of characteristic polynomial

Other methods

- ***Change of variable*** Apply substitution method here, substituting for example for $y' = f(ax + by + c)$ $u = ax + by$ to make the integral simpler. Mostly intuition-based (as is the case with integration by substitution)
- ***Separation of variables*** For equations of form $y' = a(y) \cdot b(x)$ (NOTE: Not linear), we transform into $\frac{y'}{a(y)} = b(x)$ and then integrate by substituting $y'(x)dx = dy$, changing the variable of integration. Solution: $A(y) = B(x) + c$, with $A = \int \frac{1}{a}$ and $B(x) = \int b(x)$. To get final solution, solve for the above equation for y.

3 Differential Calculus in Vector Space

3.2 Continuity

D 3.2.1: *(Convergence in $\mathbb{R}^n$)* Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$ with $x_k = (x_{k,1}, \ldots, x_{k,n})$ and let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. (x_k) converges to y as $k \to +\infty$ if $\forall\varepsilon > 0 \ \exists N \geq 1$ s.t. $\forall n \geq N$ we have $||x_k - y|| < \varepsilon$
L 3.2.2: (x_k) converges to y as $k \to +\infty$ iff one of following equiv. statements holds: ***(1)*** $\forall 1 \leq i \leq n$, the sequence $(x_{k,i})$ with $x_{k,i} \in \mathbb{R}$ converges to y_i ***(2)*** $(||x_k - y||)$ converges to 0 as $k \to +\infty$
D 3.2.3: *(Continuity)* Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. ***(1)*** Let $x_0 \in X$. f continuous in $\mathbb{R}^n$ if $\forall\varepsilon > 0 \ \exists\delta > 0$ s.t. if $x \in X$ satisfies $||x - x_0|| < \delta$, then $||f(x) - f(x_0)|| < \varepsilon$ ***(2)*** f continuous *on* X if continuous at $x_0 \ \forall x_0 \in X$ **P 3.2.4:** Let X and f as prev. Let $x_0 \in X$. f continuous at x_0 iff $\forall(x_k)_{k\geq1}$ in X s.t. $x_k \to x_0$ as $k \to +\infty$, $(f(x_k))_{k\geq1}$ in $\mathbb{R}^m$ converges to $f(x)$
D 3.2.5: *(Limit)* Let X, f and x_0 as prev. and $y \in \mathbb{R}^m$. *f has limit y as $x \to x_0$* with $x \neq x_0$ if $\forall\varepsilon > 0 \ \exists\delta > 0$ s.t. $\forall x \neq x_0 \in X, ||x - x_0|| < \delta$ we have $||f(x) - y|| < \varepsilon$. We write $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$ **R 3.2.6:** Also possible without ass. that $x_0 \in X$

P 3.2.7: Let X, f, x_0 and y as prev. We have $\lim_{\substack{x\to x_0\\ x\neq x_0}} f(x) = y$ iff $\forall (x_k)$ in X s.t. $x_k \to x$ as $k \to +\infty$ and $x_k \neq x_0$ $(f(x_k))$ in $\mathbb{R}^m$ converges to y **P 3.2.9:** Let $X \subseteq \mathbb{R}^n$, $y \subseteq \mathbb{R}^m$, $p \in \mathbb{N}$ and let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be cont. Then $g \circ f$ is continuous

Remark: To find the limits, we have two tricks (for $\lim_{(x,y)\to(a,b)}$):

1. ***(Substitution)*** Substitute $y = x + (b - a)$, then limit is $\lim_{x\to(a-b)}$
2. ***(Polar coordinates)*** Substitute $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ and the limit is $\lim_{r\to 0}$

Ex 3.2.10: ***(1)*** $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ continuous $\Rightarrow f = (f_1, f_2) : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2}$ is continuous (Cartesian product) ***(2)*** Any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. In particular, the identity map is continuous ***(3)*** If $f_1, \dots, f_n$ continuous, then $f(x_1, \dots, x_n) = f_1(x_1) \cdot \dots \cdot f_n(x_n)$ is continuous ***(4)*** Polynomials in $x_1, \dots, x_n$ are continuous ***(5)*** $f_1 f_2$ is continuous if f_1 and f_2 are continuous and if $f_2(x) \neq 0 \;\; \forall x \in X$, then $f_1 \div f_2$ is continuous. (see Theorem 2.1.8 in Analysis I)
(6) If both f and g have limits, then $\lim_{x\to x_0}(f(x)+g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$ and analogous for $\times$ ***(7)*** If $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, then $g(x) = f(x, y_0)$ for $y_0 \in \mathbb{R}$ is continuous. The converse is not true

D 3.2.11: ***(1)*** $X \subseteq \mathbb{R}^n$ is ***bounded*** if the set of $||x||$ for $x \in X$ is bounded in $\mathbb{R}$ ***(2)*** $X \subseteq \mathbb{R}^n$ is ***closed*** if $\forall (x_k)$ in X that converge in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$ ***(3)*** $X \subseteq \mathbb{R}^n$ is ***compact*** if it is bounded and closed

Ex 3.2.12: ***(1)*** $\emptyset$ and $\mathbb{R}^n$ are closed. ***(2)*** The *open* disc $D = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$ for $r > 0$ and $x_0 \in \mathbb{R}^n$ is bounded and not closed. ***(3)*** The *closed* disc $\Delta = \{x \in \mathbb{R}^n : ||x - x_0|| \leq r\}$ is bounded and closed. In particular, a closed interval is a closed set. An interval is compact if it is bounded ***(4)*** If $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^m$ are bounded (also closed or compact), then so is $X_1 \times X_2 \subseteq \mathbb{R}^{n+m}$

P 3.2.13: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed $Y \subseteq \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subseteq \mathbb{R}^n$ is closed

Ex 3.2.14: The ***zero set*** $Z = \{x \in \mathbb{R}^n : f(x) = 0\}$ is closed in $\mathbb{R}^n$ because $\{0\} \subseteq \mathbb{R}$ is closed. More generally: for any $r \geq 0$, $\{x \in \mathbb{R}^n : |f(x)| \leq r\}$ is $f^{-1}([-r, r])$ and is closed, since $[-r, r]$ is closed. Furthermore: $\{x \in \mathbb{R}^3 : ||x - x_0|| = r\}$ is closed

T 3.2.15: Let $(X \neq \emptyset) \subseteq \mathbb{R}^n$ compact and $f : X \to \mathbb{R}$ continuous. Then f bounded, has max and min, i.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup_{x\in X} f(x)$ and $f(x_-) = \inf_{x\in X} f(x)$

3.3 Partial derivatives

D 3.3.1: $X \subseteq \mathbb{R}^n$ ***open*** if for any $x = (x_1, \dots, x_n) \in X \; \exists \delta > 0$ s.t. $\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \;\; \forall i\}$ is contained in X. (= changing a coordinate of x by $< \delta \to x' \in X$) **P 3.3.2:** $X \subseteq \mathbb{R}^n$ open $\Leftrightarrow$ ***complement*** $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed

C 3.3.3: If $f : \mathbb{R}^n \to \mathbb{R}^m$ cont. and $Y \subseteq \mathbb{R}^m$ open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$ **Ex 3.3.4:** ***(1)*** $\emptyset$ and $\mathbb{R}^n$ are both open and closed. ***(2)*** Open ball $D = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$ is open in $\mathbb{R}^n$ (x_0 the center and r radius) ***(3)*** $I_1 \times \dots \times I_n$ is open in $\mathbb{R}^n$ for I_i open ***(4)*** $X \subseteq \mathbb{R}^n$ open $\Leftrightarrow \forall x \in X \exists \delta > 0$ s.t. open ball of center x and radius δ is contained in X

D 3.3.5: *(Partial derivative)* Let $X \subseteq \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ and $1 \leq i \leq n$. Then f has partial derivative on X with respect to the i-th variable (or coordinate), if $\forall x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$, $g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, x_{0,n})$ on set $I = \{t \in \mathbb{R} : (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. The derivative $g'(x_{0,i})$ at $x_{0,i}$ is denoted: $\frac{\partial f}{\partial x_i}(x_0), \partial_{x_i} f(x_0)$ or $\partial_i f(x_0)$

P 3.3.6: Let $X \subseteq \mathbb{R}^n$ open, $f, g : X \to \mathbb{R}^m$ and $1 \leq i \leq n$. Then: ***(1)*** If f & g have ∂_i on X, then so does $f + g$ and $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$ ***(2)*** If $m = 1$ (i.e. $\mathbb{R}^1$) and f & g have ∂_i on X, then so does fg and $\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g)$ and if $g(x) \neq 0 \;\; \forall x \in X$, then if $f \div g$ has ∂_i on X, then so does $f \div g$ and $\partial_{x_i}(f \div g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g)) \div g^2$