# 1 Linear Algebra

Relevant definitions used throughout Analysis II.

$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$

Def **Scalar Product** $x \cdot y := \sum_{i=0}^{n}(x_i \cdot y_i)$

Def **Euclidian Norm** $\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}$

Used to generalize $|x|$ in many Analysis I definitions

Lem. **Properties of** $\|x\|$

$(i) \quad \|x\| \geq 0$

$(ii) \quad \|x\| \iff x = 0$

$(iii) \quad \|\alpha x\| = \alpha \cdot \|x\|$

$(iv) \quad \|x + y\| \leq \|x\| + \|y\| \quad$ (Triangle Inequality)

Def **Definiteness**

Positive Definite $\quad \overset{\text{def}}{\iff} \quad x^\top \mathbf{A} x > 0 \; \forall x \in \mathbb{R}^n_{\neq 0}$

Negative Definite $\quad \overset{\text{def}}{\iff} \quad x^\top \mathbf{A} x < 0 \; \forall x \in \mathbb{R}^n_{\neq 0}$

If 0 is allowed, $\mathbf{A}$ is called positive/negative semi-definite.

Def **Trace** $\text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$

# 2 Differential Equations

Def **Differential Equation** (DE)
Equation relating unknown $f$ to derivatives $f^{(i)}$ at *same* $x$.

Def **Ordinary Differential Equation** (ODE)
DE s.t. $f : I \to \mathbb{R}$ is in one variable.

Def **Partial Differential Equation** (PDE)
DE s.t. $f : I^d \to \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def **Order** $\text{ord}(F) := \max_{i \geq 0}\{i \mid f^{(i)} \in F, \; f^{(i)} \neq 0\}$

Remark Any $F$ s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

## Solutions to ODEs

$\forall F : \mathbb{R}^2 \to \mathbb{R}$ s.t. $F$ is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$$\exists f : I \to \mathbb{R}$$
s.t. $\forall x \in I : f'(x) = F(x, f(x))$ and $f(x_0) = y_0$

s.t. $I$ is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

## 2.1 Linear Differential Equations

Def **Linear Differential Equation** (LDE)

$$y^{(k)} + a_{k-1} y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

$I \subset \mathbb{R}$ is open, $\quad k \geq 1, \quad \forall i < k : a_i : I \to \mathbb{C}$

Def Homogeneity of LDEs

**Homogeneous** $\quad \overset{\text{def}}{\iff} \quad b = 0$

**Inhomogeneous** $\quad \overset{\text{def}}{\iff} \quad b \neq 0$

Remark $D(y) := y^{(k)} + \ldots + a_0 y$ is a linear operation:

$$D(z_1 f_1 + z_2 f_2) = z_1 D(f_1) + z_2 D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$ $k$-times differentiable

Def **Homogeneous Solution Space**
$\mathcal{S}(F) := \{f : I \to \mathbb{C} \mid f$ solves $F, f$ is $k$-times diff.$\}$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: $f$ to $D(f)$:

$$D(f) = z_1 D(f_1) + z_2 D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$

## Solutions for complex homogeneous LDEs

$F$ s.t. $a_0, \ldots, a_{k-1}$ continuous and complex-valued
1. $\mathcal{S}$ is a complex vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

## Solutions for real homogeneous LDEs

$F$ s.t. $a_0, \ldots, a_{k-1}$ continuous and real-valued
1. $\mathcal{S}$ is a real vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def **Inhomogeneous Solution Space**
$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), \; f_0$ is a particular sol.$\}$
Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

## Solutions for real inhomogeneous LDEs

$F$ s.t. $a_0, \ldots, a_{k-1}$ continuous, $b : I \to \mathbb{C}$
1. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If $b, a_i$ are real-valued, a real-valued sol. exists.

Remark **Applications of Linearity**
If $f_1$ solves $F$ for $b_1$, and $f_2$ for $b_2$: $f_1 + f_2$ solves $b_1 + b_2$.
Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

# 3 Solutions to Differential Equations

## 3.1 Linear Solutions: First Order

**Form:** $y' + ay = b$ $\qquad I \subset \mathbb{R}, \quad a,b : I \to \mathbb{R}$

**Approach:**

1. Hom. Solution $f_1$ for: $y' + ay = 0$
   Note that $\mathcal{S}$ has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for $\mathcal{S}$

2. Part. Solution $f_0$ for $y' + ay = b$

**Solutions:** $f_0 + z f_1$ $\quad$ for $z \in \mathbb{C}$

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**Explicit Homogeneous Solution**

$A(x)$ is a primitive of $a$, $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$
$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

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Method **Variation of Constants**: Treating $z$ as $z(x)$ yields:

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**Explicit Inhomogeneous Solution**

$A(x)$ is a primitive of $a$

$$f_0(x) = \underbrace{\left( \int b(x) \cdot \exp(A(x)) \right)}_{z(x)} \cdot \exp\left( -A(x) \right)$$

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Method **Educated Guess**
Usually, $y$ has a similar form to $b$:

| $b(x)$ | Guess |
|---|---|
| $a \cdot e^{\alpha x}$ | $b \cdot e^{\alpha x}$ |
| $a \cdot \sin(\beta x)$ | $c \sin(\beta x) + d \cos(\beta x)$ |
| $b \cdot \cos(\beta x)$ | $c \sin(\beta x) + d \cos(\beta x)$ |
| $a e^{\alpha x} \cdot \sin(\beta x)$ | $e^{\alpha x} \left( c \sin(\beta x) + d \cos(\beta x) \right)$ |
| $b e^{\alpha x} \cdot \cos(\beta x)$ | $e^{\alpha x} \left( c \sin(\beta x) + d \cos(\beta x) \right)$ |
| $P_n(x) \cdot e^{\alpha x}$ | $R_n(x) \cdot e^{\alpha x}$ |
| $P_n(x) \cdot e^{\alpha x} \sin(\beta x)$ | $e^{\alpha x} \left( R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x) \right)$ |
| $P_n(x) \cdot e^{\alpha x} \cos(\beta x)$ | $e^{\alpha x} \left( R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x) \right)$ |

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Remark If $\alpha, \beta$ are roots of $P(X)$ with multiplicity $j$, multiply guess with a $P_j(x)$.

## 3.2 Linear Solutions: Constant Coefficients

**Form:**

$$y^{(k)} + a_{k-1} y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

Where $a_0, \ldots, a_{k-1} \in \mathbb{C}$ are constants, $b(x)$ is continuous.

### 3.2.1 Homogeneous Equations

The idea is to find a Basis of $\mathcal{S}$:

Def **Characteristic Polynomial** $P(X) = \prod_{i=1}^{k}(X - \alpha_i)$

Remark The unique roots $\alpha_1, \ldots, \alpha_l$ form a Basis:

$$\text{span}(\mathcal{S}) = \{ x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i \}$$

$v_1, \ldots, v_k$ are the Multiplicities of $\alpha_1, \ldots, \alpha_k$

Remark If $\alpha_j = \beta + \gamma i \in \mathbb{C}$ is a root, $\bar{\alpha}_j = \beta - \gamma i$ is too.
To get a real-valued solution, apply:

$$e^{\alpha_j x} = e^{\beta x} \left( \cos(\gamma x) + i \sin(\gamma x) \right)$$

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**Explicit Homogeneous Solution**

Using $\alpha_1, \ldots, \alpha_k$ from $P(X)$ s.t. $\alpha_i \neq \alpha_j$, $z_i \in \mathbb{C}$ arbitrary

$$f(x) = \prod_{i=1}^{k} z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)}(x) = \prod_{i=1}^{k} z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of $\mathcal{S}$

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Solutions exist $\forall Z = (z_1, \ldots, z_k)$ since that system's $\det(M_Z) \neq 0$.

### 3.2.2 Inhomogeneous Equations

Method **Undetermined Coefficients**: An educated guess.

1. $b(x) = c x^d \cdot e^{\alpha x} \implies f_p(x) = Q(x) e^{\alpha x}$
   $\deg(Q) \leq d + v_\alpha$, where $v_\alpha$ is $\alpha$'s multiplicity in $P(X)$

2. $\left. \begin{array}{l} b(x) = c x^d \cdot \cos(\alpha x) \\ b(x) = c x^d \cdot \sin(\alpha x) \end{array} \right\} f_p = Q_1(x) \cos(\alpha x) + Q_2(x) \sin(\alpha x$
   $\deg(Q_{1,2}) \leq d + v_\alpha$, where $v_\alpha$ is $\alpha$'s multiplicity in $P(X)$

Remark **Applying Linearity**
If $b(x) = \sum_{i=1}^{n} b_i(x)$, A solution for $b(x)$ is $f(x) = \sum_{i=1}^{n} f_i(x)$
Sometimes called *Superposition Principle* in this context

## 3.3 Other Methods

Method **Change of Variable**
If $f(x)$ is replaced by $h(y) = f(g(y))$, then $h$ is a sol. too.
Changes like $h(t) = f(e^t)$ may lead to useful properties.

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**Separation of Variables**

Form:
$$y' = a(y) \cdot b(x)$$

Solve using:

$$\int \frac{1}{a(y)} \, \mathrm{d}y = \int b(x) \, \mathrm{d}x + c$$

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Usually $\int 1/a(y) \, \mathrm{d}y$ can be solved directly for $\ln |a(y)| + c$.

## 3.4 Method Overview

| Method | Use case |
|---|---|
| Variation of constants | LDE with $\text{ord}(F) = 1$ |
| Characteristic Polynomial | Hom. LDE w/ const. coeff. |
| Undetermined Coefficients | Inhom. LDE w/ const. coeff. |
| Separation of Variables | ODE s.t. $y' = a(y) \cdot b(x)$ |
| Change of Variables | e.g. $y' = f(ax + by + c)$ |

# 4 Continuous functions in $\mathbb{R}^n$

Treating functions $f : X \subset \mathbb{R}^n \to \mathbb{R}/\mathbb{C}/\mathbb{R}^m, \quad m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \to \mathbb{R}^m$ means:
$x = (x_1, \ldots, x_n), \quad f(x) = f(f_1(x), \ldots, f_m(x))$

## 4.1 Multivariate functions

Def **Linear map** $f : \mathbb{R}^n \to \mathbb{R}^m$
In other words: $f(x) = \mathbf{A}x, \quad \mathbf{A} \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

Def **Affine Linear map** $f(x) \mapsto \mathbf{A}x + c$

Def **Quadratic form** $Q : \mathbb{R}^n \to \mathbb{R}$
In other words: $Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} (a_{i,j} x_i x_j)$

Def **Monomials** $M(x) : \mathbb{R}^n \to \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$
For example: $f(x,y,z) = 16x^2 y z^5$

Def $\deg(M) := e = \sum_{i=1}^{n} d_i$
For example: $\deg(16x^2 y z^5) = 8$

Def **Polynomials** $P(x) := \sum_{i=0}^{n} M_i(x)$
For example: $P(x,y,z) = x^3 + 25x^2 y^6 z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$
For example: $\deg(x^3 + 25x^2 y^6 z + xy) = 9$

Visualisations for some function types:

Def **Graph** $G_f := \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\}$
Only for $f : \mathbb{R}^2 \to \mathbb{R}$. Visually, this is a surface in $\mathbb{R}^3$

Def **Vector Plots** for $f : \mathbb{R}^2 \to \mathbb{R}^2$
Points in $(x,y) \in \mathbb{R}^2$ are displayed as vectors $f(x,y)$

## 4.2 Sequences in $\mathbb{R}^n$

Def **Sequences in $\mathbb{R}^n$**
$(x_k)_{k \geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \ldots x_{k,n})$

Def **Convergence in $\mathbb{R}^n$**

$$\lim_{k \to \infty} \left( x_k \right) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \quad \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

Lem. **Equivalent conditions to Convergence**

$(i) \quad \forall i \text{ s.t. } 1 \leq i \leq n : \quad \lim_{k \to \infty} \left( x_{k,i} \right) = y_i$

$(ii) \quad \lim_{k \to \infty} \left\| x_k - y \right\| = 0$

Def **Limits at points**

$$\lim_{x \neq x_0 \to x_0} \left( f(x) \right) = y \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$$

$$\forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m, \quad x_0 \in X, \quad y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

## 4.3 Continuity in $\mathbb{R}^n$

Def **Continuity in $\mathbb{R}^n$**
$f$ continuous at $x_0 \in X \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

$f$ continuous $\overset{\text{def}}{\iff} \forall x \in X : f$ continuous at $x$
$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. **Continuitiy using Sequences**
$f$ continuous at $x_0$ if and only if:

$$\forall (x_k)_{k \geq 1} : \quad \lim_{k \to \infty} \left( x_k \right) = x_0 \implies \lim_{k \to \infty} \left( f(x_k) \right) = f(x_0)$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. **Continuity of Compositions**
$f : X \to Y, \ g : Y \to \mathbb{R}^p$ continuous $\implies g \circ f$ continuous
$X \subset \mathbb{R}^n, \quad Y \subset \mathbb{R}^m, \quad p \geq 1$

Lem. **Continuity using Coordinate Functions**
$f : \mathbb{R}^n \to \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

## 4.4 Subsets of $\mathbb{R}^n$

Def **Bounded**
$X \subset \mathbb{R}^n$ bounded $\overset{\text{def}}{\iff} \left\{ \|x\| \mid x \in X \right\} \subset \mathbb{R}$ bounded.
Example: The open disc $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is bounded.

Def **Closed**
$X \subset \mathbb{R}^n$ closed $\overset{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \quad \lim_{x \to \infty} \left( x_k \right) \in X$
Example: $\emptyset, \mathbb{R}^n$ are closed.

Def **Compact** if closed and bounded.
Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is compact.

Def **Open**
$X \subset \mathbb{R}^n$ open $\overset{\text{def}}{\iff} \forall x \in X, \ \exists \delta > 0 :$

$$\left\{ y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \ \forall i \leq n \right\} \subset X$$

In other words: Changing any coord. $x_i$ by $\delta$ keeps $x'$ in $X$
Example: $\emptyset, \mathbb{R}^n$ are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. **Continous functions preserve closed/open**

$\forall$ closed/open $Y$ :

$$f^{-1}(Y) = \left\{ x \in \mathbb{R}^n \mid f(x) \in Y \right\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $\quad Y \subset \mathbb{R}^m$

Lem. **The complement of open sets is closed**

$$X \subset \mathbb{R}^n \text{ is open } \iff \underbrace{\left\{ x \in \mathbb{R}^n \mid x \notin X \right\}}_{\text{Complement}} \text{ is closed}$$

**Min-Max Theorem**

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \to \mathbb{R}$:

$$\exists x_1, x_2 \in X : \quad f(x_1) = \sup_{x \in X} f(x), \quad f(x_2) = \inf_{x \in X} f(x)$$

# 5 Differential Calculus in $\mathbb{R}^n$

## 5.1 Partial Derivatives

### Partial Derivative

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad 1 \le i \le n, \quad x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \ldots, t, \ldots, x_{0,n}) \in X\} \to \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,i}, \ldots, x_{0,t-1}, \ t\ , x_{0,t+1}, \ldots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \to t}$$

**Notation** $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

**Lem. Properties of Partial Derivatives**

Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

$(i) \quad \partial x_i(f + g) = \partial x_i f + \partial x_i g$

$(ii) \quad \partial x_i(fg) = \partial x_i(f)g + \partial x_i(g)f \quad$ if $m = 1$

$(iii) \quad \partial x_i\left(\dfrac{f}{g}\right) = \dfrac{\partial x_i(f)g - \partial x_i(g)f}{g^2} \quad$ if $g(x) \ne 0 \ \forall x \in X$

$X \subset \mathbb{R}^n$ open, $\quad f.g : X \to \mathbb{R}^n, \quad 1 \le i \le n$

### The Jacobian

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^n$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial x_1 f_1(x) & \partial x_2 f_1(x) & \cdots & \partial x_n f_1(x) \\ \partial x_1 f_2(x) & \partial x_2 f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_1 f_n(x) & \partial x_2 f_n(x) & \cdots & \partial x_n f_m(x) \end{bmatrix}$$

Think of $f$ as a vector of $f_i$, then $\mathbf{J}_f$ is that vector stretched for all $x_j$

**Def Gradient** $\nabla f(x_0) := \begin{bmatrix} \partial x_1 f(x_0) \\ \vdots \\ \partial x_n f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}$, i.e. *must* map to 1 dimension

**Remark** $\nabla f$ points in the direction of greatest increase.

This generalizes that in $\mathbb{R}$, $\text{sgn}(f)$ shows if $f$ increases/decreases

**Def Divergence** $\text{div}(f)(x_0) := \text{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^n, \quad \mathbf{J}_f$ exists

## 5.2 The Differential

Partial derivatives don't provide a good approx. of $f$, unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in $\mathbb{R}^n$.

### Differentiability in $\mathbb{R}^n$ & the Differential

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^n, \quad u : \mathbb{R}^n \to \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If $f$ is differentiable at $x_0 \in X$ with $u$ s.t.

$$\lim_{x \ne x_0 \to x_0} \frac{1}{\|x - x_0\|}\left(f(x) - f(x_0) - u(x - x_0)\right) = 0$$

Similarly, $f$ is differentiable if this holds for all $x \in X$

**Lem. Properties of Differentiable Functions**

$(i) \quad$ Continuous on $X$

$(ii) \quad \forall i \le m, j \le n : \quad \partial_{x_j} f_i$ exists

$(iii) \quad m = 1 : \quad \partial_{x_i} f(x_0) = a_i$

$\qquad$ for: $\quad u(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m$ differentiable on $X$

**Lem. Preservation of Differentiability**

$(i) \quad f + g$ is differentiable: $d(f + g) = df + dg$

$(ii) \quad fg$ is differentiable, if $m = 1$

$(iii) \quad \dfrac{f}{g}$ is differentiable, if $m = 1, \ g(x) \ne 0 \ \forall x \in X$

$X \subset \mathbb{R}^n$ open, $\quad f, g : X \to \mathbb{R}^m$ differentiable on $X$

**Lem. Cont. Partial Derivatives imply Differentiability**

if all $\partial_{x_j} f_i$ exist and are continuous:

$$f \text{ differentiable on } X, \quad df(x_0) = \mathbf{J}_f(x_0)$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m$

**Lem. Chain Rule** $\quad g \circ f$ is differentiable on $X$

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ \mathbf{J}_{g \circ f}(x_0) &= \mathbf{J}_g(f(x_0)) \cdot \mathbf{J}_f(x_0) \end{aligned}$$

$X \subset \mathbb{R}^n$ open, $\quad Y \subset \mathbb{R}^m$ open, $\quad f : X \to Y, g : Y \to \mathbb{R}^p, f, g$ diff.-able

**Def Tangent Space**

$$T_f(x_0) := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\right\}$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m$ diff.-able, $\quad x_0 \in X, \quad u = df(x_0)$

**Def Directional Derivative**

$$D_v f(x_0) = \lim_{t \ne 0 \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m, \quad v \ne 0 \in \mathbb{R}^n, \quad x_0 \in X$

**Lem. Directional Derivatives for Diff.-able Functions**

$$D_v f(x_0) = df(x_0)(v) = \mathbf{J}_f(x_0) \cdot v$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m$ diff.-able, $\quad v \ne 0 \in \mathbb{R}^n, \quad x_0 \in X$

**Remark** $D_v f$ is linear w.r.t $v$, so: $D_{v_1 + v_2} f = D_{v_1} f + D_{v_2} f$

**Remark** $D_v f(x_0) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$

In the case $f : X \to \mathbb{R}$, where $\theta$ is the angle between $v$ and $\nabla f(x_0)$

## 5.3 Higher Derivatives

Def **Differentiability Classes**

$f \in C^1(X; \mathbb{R}^m) \quad \overset{\text{def}}{\iff} \quad f$ diff.-able on $X$, all $\partial_{x_j} f_i$ exist

$f \in C^k(X; \mathbb{R}^m) \quad \overset{\text{def}}{\iff} \quad f$ diff.-able on $X$, all $\partial_{x_j} f_i \in C^{k-1}$

$f \in C^\infty(X; \mathbb{R}^m) \quad \overset{\text{def}}{\iff} \quad f \in C^k(X; \mathbb{R}^m) \ \forall k \geq 1$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m$

Lem. Polynomials, Trig. functions and $\exp$ are in $C^\infty$

Lem. **Operations preserve Differentiability Classes**

$$
\begin{aligned}
(i) & \quad f + g \quad \in C^k \\
(ii) & \quad fg \quad \in C^k \quad \text{if } m = 1 \\
(iii) & \quad \frac{f}{g} \quad \in C^k \quad \text{if } m = 1, g(x) \neq 0 \ \forall x \in X
\end{aligned}
$$

$f, g \in C^k$

Lem. **Composition preserves Differentiability Classes**

$$g \circ f \in C^k$$

$f \in C^k, \quad f(X) \subset Y, \quad Y \subset \mathbb{R}^m$ open, $\quad g : Y \to \mathbb{R}^p, \quad g \in C^k$

**Partial Derivatives commute in $C^k$**

$k \geq 2, \quad X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}^m, \quad f \in C^k$

$$\forall x, y : \quad \partial_{x,y} f = \partial_{y,x} f$$

This generalizes for $\partial_{x_1, \ldots, x_n} f$.

Remark Linearity of Partial Derivatives

$$\partial_x^m (a f_1 + b f_2) = a \partial_x^m f_1 + b \partial_x^m f_2$$

Assuming both $\partial_x f_{1,2}$ exist.

Def **Laplace Operator**

$$\Delta f := \text{div}\big(\nabla f(x)\big) = \sum_{i=0}^n \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right) = \sum_{i=0}^n \frac{\partial^2 f}{\partial x_i^2}$$

---

**The Hessian**

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$

$$
\mathbf{H}_f(x) := \begin{bmatrix}
\partial_{1,1} f(x_0) & \partial_{2,1} f(x_0) & \cdots & \partial_{n,1} f(x_0) \\
\partial_{1,2} f(x_0) & \partial_{2,2} f(x_0) & \cdots & \partial_{n,2} f(x_0) \\
\vdots & \vdots & \ddots & \vdots \\
\partial_{1,n} f(x_0) & \partial_{2,n} f(x_0) & \cdots & \partial_{n,n} f(x_0)
\end{bmatrix}
$$

Where $\big( \mathbf{H}_f(x) \big)_{i,j} = \partial_{x_i, x_j} f(x)$

Note that $f : X \to \mathbb{R}$, i.e. $\mathbf{H}_f$ only exists for 1-dimensionally valued $f$

Notation $\mathbf{H}_f(x) = \text{Hess}_f(x) = \nabla^2 f(x)$

Remark $\mathbf{H}_f(x_0)$ is symmetric: $\big( \mathbf{H}_f(x_0) \big)_{i,j} = \big( \mathbf{H}_f(x_0) \big)_{j,i}$

Def **Polar Coordinates**

$$g(r, \theta) = \big( r \cos(\theta), r \sin(\theta) \big)$$

$$
\mathbf{J}_g(r, \theta) = \begin{bmatrix}
\cos(\theta) & -r \sin(\theta) \\
\sin(\theta) & r \cos(\theta)
\end{bmatrix}
$$

$$\partial_x f = \cos(\theta) \partial_r f - \frac{1}{r} \sin(\theta) \partial_\theta f$$

$$\partial_y f = \sin(\theta) \partial_r f + \frac{1}{r} \cos(\theta) \partial_\theta f$$

$(r, \theta) \in (0, +\infty) \times \mathbb{R}, \quad \det(\mathbf{J}_g) = r$

## 5.4 Taylor Polynomials

Def $|m| := \sum_{i=1}^n m_1$ $\qquad$ Def $y^m := y_1^m \cdots y_n^m$

Def $m! := m_1! \cdots m_n!$

for $m = (m_1, \ldots, m_n), \quad y = (y_1, \ldots, y_n)$

**Taylor Polynomials**

$k \geq 1, \quad f : X \to \mathbb{R}, \quad f \in C^k, \quad x_0 \in X$

$$T_k f(y; x_0) := \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

---

Lem. **Taylor Approximation**

$$\lim_{x \neq x_0 \to x_0} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

Where $f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$

$k \geq 1, \quad X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^k, \quad x_0 \in X$

Remark Taylor polynomials of degree $1, 2$:

$$T_1 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} \Big( x_0^\top \cdot \mathbf{H}_f(y) \cdot x_0 \Big)$$

Method Calculating $T_k f(y; x_0)$ also yields $\mathbf{H}_f$ for $k \geq 2$.

$$T_2 f((x_0, y_0); (x, y)) = \ldots + a x^2 + b y^2 + c x y$$

$$\implies \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} 2a & c \\ c & 2b \end{bmatrix}$$

Method Taylor Polynomials can be found by combination.

**Example:** $f(x, y) = \underbrace{e^{y^4}}_{1} + \underbrace{\sin(xy)}_{2} + \underbrace{2xy^2}_{3} - \underbrace{\ln(x^2 + 1)}_{4}, \quad k = 3$

1. $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \implies e^{y^4} \approx 1 + y^4 + \frac{y^8}{2} + \frac{y^{12}}{6}$
   Since $k = 3$, discarding all terms with $\deg > 3$ yields: $e^{y^3} \approx 1$
2. $\sin(x) \approx x - \frac{x^3}{6} \implies \sin(xy) \approx xy$
3. $2xy^2 \approx 2xy^2$ $\quad$ (Since it's already a polynomial, $\deg = 3$)
4. $\ln(x + 1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \implies \ln(x^2 + 1) \approx x^2$

Thus: $f(x) \approx 1 + xy + 2xy^2 - x^2 = T_3 f\big((0, 0); (x, y)\big)$

## 5.5 Critical Points

Lem. **Local Maxima & Minima**

$$\left.\begin{array}{l} f(y) \leq f(x_0) \ \forall y \ \text{close} \\ f(y) \geq f(x_0) \ \forall y \ \text{close} \end{array}\right\} \quad \frac{\partial f}{\partial x_i}(x_0) = 0 \ \ \forall i \leq n$$

In other words: $df(x_0) = \nabla f(x_0) = 0$
$f : X \to \mathbb{R}, \quad X \subset \mathbb{R}^n$ open, $f$ diff.-able

Def **Critical Point**

$$x_0 \in X \text{ is critical} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \nabla f(x_0) = 0$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$ diff.-able

Remark **Existance of Maxima/Minima**
Don't *have to* exist if $X$ is open, only if $X$ is compact.
However, for compact sets, the lemma above no longer applies.

Method **Critical points on Compact Sets**
Decompose $X = X' \cup B$, s.t. $X'$ is open, $B$ is a *boundary*.

1. Find critical points in $X'$

2. Check if any $x \in B$ is a maximum/minimum

Def **Non-degenerate Critical Point**

$$x_0 \in X \text{ non-deg.} \overset{\text{def}}{\Longleftrightarrow} \det\Big(\mathbf{H}_f(x_0)\Big) \neq 0$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$ is critical
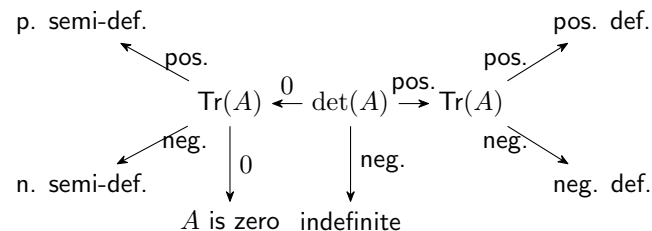
Lem. **Definiteness of the Hessian**

$\mathbf{H}_f(x_0)$ positive definite $\qquad \Longrightarrow \ x_0$ is a local min.
$\mathbf{H}_f(x_0)$ negative definite $\qquad \Longrightarrow \ x_0$ is a local max.
$\mathbf{H}_f(x_0)$ indefinite $\qquad\qquad \Longrightarrow \ x_0$ is a saddle point.

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$ non-deg. critical

Method **Determining Definiteness for $2 \times 2$ Matrices**

p. semi-def.           pos. def.

pos.        pos.

$$\mathsf{Tr}(A) \overset{0}{\leftarrow} \det(A) \overset{\text{pos.}}{\to} \mathsf{Tr}(A)$$

neg.    $0$    neg.    neg.

n. semi-def.           neg. def.

$A$ is zero   indefinite

# 6 Integral Calculus in $\mathbb{R}^n$

## 6.1 Line Integrals

**Integrals for $f : I \to \mathbb{R}^n$**

$I = [a, b]$ closed & bounded, $\quad f : I \to \mathbb{R}^n$ cont.

$$\int_a^b f(t) \ dt = \left( \int_a^b f_1(t) \ dt, \dots, \int_a^b f_n(t) \ dt \right)$$

Def **Piecewise Continuity**
$\exists k \geq 1$, and a Partition $a = t_0 < \cdots < t_k = b$
s.t. $f_j : [t_{j-1}, t_j] \to \mathbb{R}^n$ has $f_j \in C^1$ for all $j \leq k$
For $f : I \to \mathbb{R}^n$

Def **Parametrized Curve** $\gamma : [a, b] \to \mathbb{R}^n$ pw.-cont.
Also called *Path* from $\gamma(a)$ to $\gamma(b)$

**Line Integral**

$\gamma : [a, b] \to \mathbb{R}^n$ is path, $\quad X \subset \mathbb{R}^n$ s.t. $\gamma([a, b]) \subset X$
$f : X \to \mathbb{R}^n$ continuous

$$\int_\gamma f(s) \cdot \ ds := \int_a^b f\Big(\gamma(t)\Big) \cdot \gamma'(t) \ dt$$

Def **Continuous integrals are linear**

$$\int_a^b \Big( f(t) + g(t) \Big) \ dt = \int_a^b f(t) \ dt + \int_a^b g(t) \ dt$$

$f, g : I \to \mathbb{R}^n$ continuous

Remark $f : X \to \mathbb{R}^n$ is called a *Vector Field*.

Def **Oriented Reparametrization**