1 Linear Algebra

Relevant definitions used throughout Analysis II.

$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$

Def **Euclidian Norm** $||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$

Used to generalize $|x|$ in many Analysis I definitions

Lem. **Properties of $||x||$**

$(i) \quad ||x|| \geq 0$

$(ii) \quad ||x|| \iff x = 0$

$(iii) \quad ||\alpha x|| = \alpha \cdot ||x||$

$(iv) \quad ||x + y|| \leq ||x|| + ||y||$ (Triangle Inequality)

Def **Trace** $\text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$

2 Differential Equations

Def **Differential Equation** (DE)
Equation relating unknown f to derivatives $f^{(i)}$ at *same* x.

Def **Ordinary Differential Equation** (ODE)
DE s.t. $f : I \to \mathbb{R}$ is in one variable.

Def **Partial Differential Equation** (PDE)
DE s.t. $f : I^d \to \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def **Order** $\text{ord}(F) := \max_{i \geq 0}\{i \mid f^{(i)} \in F, \ f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \to \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$\exists f : I \to \mathbb{R}$
s.t. $\forall x \in I : f'(x) = F(x, f(x))$ and $f(x_0) = y_0$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

2.1 Linear Differential Equations

Def **Linear Differential Equation** (LDE)

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

$I \subset \mathbb{R}$ is open, $\quad k \geq 1, \quad \forall i < k : a_i : I \to \mathbb{C}$

Def Homogeneity of LDEs

Homogeneous $\overset{\text{def}}{\iff} b = 0$

Inhomogeneous $\overset{\text{def}}{\iff} b \neq 0$

Remark $D(y) := y^{(k)} + \ldots + a_0 y$ is a linear operation:

$$D(z_1 f_1 + z_2 f_2) = z_1 D(f_1) + z_2 D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$ k-times differentiable

Def **Homogeneous Solution Space**
$\mathcal{S}(F) := \{f : I \to \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1 D(f_1) + z_2 D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$

Solutions for complex homogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous and complex-valued

1. $\mathcal{S}$ is a complex vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous and real-valued

1. $\mathcal{S}$ is a real vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def **Inhomogeneous Solution Space**
$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), \ f_0 \text{ is a particular sol.}\}$
Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. $a_0, \ldots, a_{k-1}$ continuous, $b : I \to \mathbb{C}$

1. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark **Applications of Linearity**
If f_1 solves F for b_1, and f_2 for b_2: $f_1 + f_2$ solves $b_1 + b_2$.
Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

3 Solutions to Differential Equations

3.1 Linear Solutions: First Order

Form: $y' + ay = b \qquad I \subset \mathbb{R}, \quad a, b : I \to \mathbb{R}$

Approach:

1. Hom. Solution f_1 for: $y' + ay = 0$
   Note that $\mathcal{S}$ has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for $\mathcal{S}$
2. Part. Solution f_0 for $y' + ay = b$

Solutions: $f_0 + zf_1$ for $z \in \mathbb{C}$

Explicit Homogeneous Solution

$A(x)$ is a primitive of a, $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$
$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

Method **Variation of Constants**: Treating z as $z(x)$ yields:

Explicit Inhomogeneous Solution

$A(x)$ is a primitive of a

$$f_0(x) = \underbrace{\left(\int b(x) \cdot \exp(A(x))\right)}_{z(x)} \cdot \exp\left(-A(x)\right)$$

Method **Educated Guess**
Usually, y has a similar form to b:

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \cdot \sin(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$b \cdot \cos(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$ae^{\alpha x} \cdot \sin(\beta x)$	$e^{\alpha x}\left(c\sin(\beta x) + d\cos(\beta x)\right)$
$be^{\alpha x} \cdot \cos(\beta x)$	$e^{\alpha x}\left(c\sin(\beta x) + d\cos(\beta x)\right)$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$
$P_n(x) \cdot e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$

Remark If α, β are roots of $P(X)$ with multiplicity j, multiply guess with a $P_j(x)$.

3.2 Linear Solutions: Constant Coefficients

Form:

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

Where $a_0, \ldots, a_{k-1} \in \mathbb{C}$ are constants, $b(x)$ is continuous.

3.2.1 Homogeneous Equations

The idea is to find a Basis of $\mathcal{S}$:

Def **Characteristic Polynomial** $P(X) = \prod_{i=1}^{k}(X - \alpha_i)$

Remark The unique roots $\alpha_1, \ldots, \alpha_l$ form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}$$

$v_1, \ldots, v_k$ are the Multiplicities of $\alpha_1, \ldots, \alpha_k$

Remark If $\alpha_j = \beta + \gamma i \in \mathbb{C}$ is a root, $\bar{\alpha_j} = \beta - \gamma i$ is too. To get a real-valued solution, apply:

$$e^{\alpha_j x} = e^{\beta x}\left(\cos(\gamma x) + i\sin(\gamma x)\right)$$

Explicit Homogeneous Solution

Using $\alpha_1, \ldots, \alpha_k$ from $P(X)$ s.t. $\alpha_i \neq \alpha_j$, $z_i \in \mathbb{C}$ arbitrary

$$f(x) = \prod_{i=1}^{k} z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)(x)} = \prod_{i=1}^{k} z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of $\mathcal{S}$

Solutions exist $\forall Z = (z_1, \ldots, z_k)$ since that system's $\det(M_Z) \neq 0$.

3.2.2 Inhomogeneous Equations

Method **Undetermined Coefficients**: An educated guess.

1. $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$
   $\deg(Q) \leq d + v_\alpha$, where v_α is α's multiplicity in $P(X)$
2. $\left.\begin{array}{l} b(x) = cx^d \cdot \cos(\alpha x) \\ b(x) = cx^d \cdot \sin(\alpha x) \end{array}\right\} f_p = Q_1(x)\cos(\alpha x) + Q_2(x)\sin(\alpha x$
   $\deg(Q_{1,2}) \leq d + v_\alpha$, where v_α is α's multiplicity in $P(X)$

Remark **Applying Linearity**
If $b(x) = \sum_{i=1}^{n} b_i(x)$, A solution for $b(x)$ is $f(x) = \sum_{i=1}^{n} f_i(x)$
Sometimes called *Superposition Principle* in this context

3.3 Other Methods

Method **Change of Variable**
If $f(x)$ is replaced by $h(y) = f(g(y))$, then h is a sol. too.
Changes like $h(t) = f(e^t)$ may lead to useful properties.

Separation of Variables

Form:
$$y' = a(y) \cdot b(x)$$
Solve using:
$$\int \frac{1}{a(y)}\,\mathrm{d}y = \int b(x)\,\mathrm{d}x + c$$

Usually $\int 1/a(y)\,\mathrm{d}y$ can be solved directly for $\ln|a(y)| + c$.

3.4 Method Overview

Method	Use case
Variation of constants	LDE with ord$(F) = 1$
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. $y' = a(y) \cdot b(x)$
Change of Variables	e.g. $y' = f(ax + by + c)$

4 Continuous functions in $\mathbb{R}^n$

Treating functions $f : X \subset \mathbb{R}^n \to \mathbb{R}/\mathbb{C}/\mathbb{R}^m, \quad m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \to \mathbb{R}^m$ means:
$x = (x_1, \dots, x_n), \quad f(x) = f\big(f_1(x), \dots, f_m(x)\big)$

4.1 Multivariate functions

Def Linear map $f : \mathbb{R}^n \to \mathbb{R}^m$
In other words: $f(x) = \mathbf{A}x, \quad \mathbf{A} \in \mathbb{C}^{m\times n}$

Linear Maps are continuous

Def Affine Linear map $f(x) \mapsto \mathbf{A}x + c$

Def Quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$
In other words: $Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} (a_{i,j} x_i x_j)$

Def Monomials $M(x) : \mathbb{R}^n \to \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$
For example: $f(x, y, z) = 16x^2yz^5$

Def $\deg(M) := e = \sum_{i=1}^{n} d_i$
For example: $\deg(16x^2yz^5) = 8$

Def Polynomials $P(x) := \sum_{i=0}^{n} M_i(x)$
For example: $P(x, y, z) = x^3 + 25x^2y^6z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$
For example: $\deg(x^3 + 25x^2y^6z + xy) = 9$

Visualisations for some function types:

Def Graph $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$
Only for $f : \mathbb{R}^2 \to \mathbb{R}$. Visually, this is a surface in $\mathbb{R}^3$

Def Vector Plots for $f : \mathbb{R}^2 \to \mathbb{R}^2$
Points in $(x, y) \in \mathbb{R}^2$ are displayed as vectors $f(x, y)$

4.2 Sequences in $\mathbb{R}^n$

Def Sequences in $\mathbb{R}^n$
$(x_k)_{k\geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \dots x_{k,n})$

Def Convergence in $\mathbb{R}^n$

$$\lim_{k\to\infty} \Big(x_k\Big) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \quad ||x_k - y|| < \epsilon$$

Using this definition preserves many familiar results:

Lem. Equivalent conditions to Convergence
$(i) \quad \forall i$ s.t. $1 \leq i \leq n : \quad \lim_{k\to\infty} \Big(x_{k,i}\Big) = y_i$
$(ii) \quad \lim_{k\to\infty} \Big\| x_k - y \Big\| = 0$

Def Limits at points

$$\lim_{x\neq x_0 \to x_0} \Big(f(x)\Big) = y \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$$
$$\forall x \neq x_0 \in X : \big\|x - x_0\big\| < \delta \implies \big\|f(x) - y\big\| < \epsilon$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m, \quad x_0 \in X, \quad y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

4.3 Continuity in $\mathbb{R}^n$

Def Continuity in $\mathbb{R}^n$
f continuous at $x_0 \in X \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\big\|x - x_0\big\| < \delta \implies \big\|f(x) - f(x_0)\big\| < \epsilon$$

f continuous $\overset{\text{def}}{\iff} \forall x \in X : f$ continuous at x
$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. Continuitiy using Sequences
f continuous at x_0 if and only if:

$$\forall (x_k)_{k\geq 1} : \quad \lim_{k\to\infty} \Big(x_k\Big) = x_0 \implies \lim_{k\to\infty} \Big(f(x_k)\Big) = f(x_0)$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. Continuity of Compositions
$f : X \to Y,\ g : Y \to \mathbb{R}^p$ continuous $\implies g \circ f$ continuous
$X \subset \mathbb{R}^n, \quad Y \subset \mathbb{R}^m, \quad p \geq 1$

Lem. Continuity using Coordinate Functions
$f : \mathbb{R}^n \to \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

4.4 Subsets of $\mathbb{R}^n$

Def Bounded
$X \subset \mathbb{R}^n$ bounded $\overset{\text{def}}{\iff} \Big\{\big\|x\big\| \mid x \in X\Big\} \subset \mathbb{R}$ bounded.
Example: The open disc $D = \{x \in \mathbb{R}^n \mid \big\|x - x_0\big\| < r\}$ is bounded.

Def Closed
$X \subset \mathbb{R}^n$ closed $\overset{\text{def}}{\iff} \forall (x_k)_{k\geq 1} \in X : \quad \lim_{x\to\infty} \Big(x_k\Big) \in X$
Example: $\emptyset$, $\mathbb{R}^n$ are closed.

Def Compact if closed and bounded.
Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \big\|x - x_0\big\| \leq r\}$ is compact.

Def Open
$X \subset \mathbb{R}^n$ open $\overset{\text{def}}{\iff} \forall x \in X,\ \exists \delta > 0 :$

$$\big\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \quad \forall i \leq n\big\} \subset X$$

In other words: Changing any coord. x_i by δ keeps x' in X
Example: $\emptyset, \mathbb{R}^n$ are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. Continous functions preserve closed/open

$\forall$ closed/open Y :

$$f^{-1}(Y) = \big\{x \in \mathbb{R}^n \mid f(x) \in Y\big\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $\quad Y \subset \mathbb{R}^m$

Lem. The complement of open sets is closed

$$X \subset \mathbb{R}^n \text{ is open} \iff \underbrace{\big\{x \in \mathbb{R}^n \mid x \notin X\big\}}_{\text{Complement}} \text{ is closed}$$

Min-Max Theorem

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \to \mathbb{R}$:

$$\exists x_1, x_2 \in X : \quad f(x_1) = \sup_{x\in X} f(x), \quad f(x_2) = \inf_{x\in X} f(x)$$

5 Differential Calculus in $\mathbb{R}^n$

5.1 Partial Derivatives

Partial Derivative

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$, $1 \le i \le n$, $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, \ t \ , \dots, x_{0,n}) \in X\} \to \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,i}, \dots, x_{0,t-1}, \ t \ , x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \to t}$$

Notation $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

Lem. **Properties of Partial Derivatives**
Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

$$(i) \quad \partial x_i(f+g) = \partial x_i f + \partial x_i g$$
$$(ii) \quad \partial x_i(fg) = \partial x_i(f)g + \partial x_i(g)f \quad \text{if } m = 1$$
$$(iii) \quad \partial x_i\left(\frac{f}{g}\right) = \frac{\partial x_i(f)g - \partial x_i(g)f}{g^2} \quad \text{if } g(x) \neq 0 \ \forall x \in X$$

$X \subset \mathbb{R}^n$ open, $f.g : X \to \mathbb{R}^n$, $1 \le i \le n$

The Jacobian

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial x_1 f_1(x) & \partial x_2 f_1(x) & \cdots & \partial x_n f_1(x) \\ \partial x_1 f_2(x) & \partial x_2 f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_1 f_n(x) & \partial x_2 f_n(x) & \cdots & \partial x_n f_m(x) \end{bmatrix}$$

Think of f as a vector of f_i, then $\mathbf{J}_f$ is that vector stretched for all x_j

Def **Gradient** $\nabla f(x_0) := \begin{bmatrix} \partial x_1 f(x_0) \\ \vdots \\ \partial x_n f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$

Def **Divergence** $\text{div}(f)(x_0) := \text{Tr}\big(\mathbf{J}_f(x_0)\big)$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$, $\mathbf{J}_f$ exists

5.2 The Differential

Partial derivatives don't provide a good approx. of f, unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in $\mathbb{R}^n$.

Differentiability in $\mathbb{R}^n$ & the Differential

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If f is differentiable at $x_0 \in X$ with u s.t.

$$\lim_{x \neq x_0 \to x_0} \frac{1}{\lVert x - x_0 \rVert} \Big(f(x) - f(x_0) - u(x - x_0) \Big) = 0$$