Analysis II

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TITLE PAGE COMING SOON

"*Some funny quote from the lecture still needed*"
- Özlem Imamoglu, 2025

HS2025, ETHZ

Cheat-Sheet based on Lecture notes and Script

https://metaphor.ethz.ch/x/2025/hs/401-0213-16L/sc/script-analysis-II.pdf

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1 Introduction

This Cheat-Sheet does not serve as a replacement for solving exercises and getting familiar with the content. There is no guarantee that the content is 100% accurate, so use at your own risk. If you discover any errors, please open an issue or fix the issue yourself and then open a Pull Request here:

`https://github.com/janishutz/eth-summaries`

This Cheat-Sheet was designed with the HS2025 page limit of 10 A4 pages in mind. Thus, the whole Cheat-Sheet can be printed full-sized, if you exclude the title page, contents and this page. You could also print it as two A5 pages per A4 page and also print the Analysis I summary in the same manner, allowing you to bring both to the exam

2 Differential Equations

2.1 Introduction

Ex 2.1.1: $f'(x) = f(x)$ has only solution $f(x) = ae^x$ for any $a \in \mathbb{R}$; $f' - a = 0$ has only solution $f(x) = \int_{x_0}^{x} a(t)\ \mathrm{d}t$

T 2.1.6: Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a differential function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. The Ordinary Differential Equation (ODE) $y' = F(x, y)$ has a unique solution f defined on a "largest" interval I that contains x_0 such that $y_0 = f(x_0)$

2.2 Linear Differential Equations

An ODE is considered linear if and only if the ys are only scaled and not part of powers.
D 2.2.1: *(Linear differential equation of order k)* (order = highest derivative) $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$, with a_i and b functions in x. If $b(x) = 0\ \ \forall x$, ***homogeneous***, else ***inhomogeneous***
T 2.2.2: For open $I \subseteq \mathbb{R}$ and $k \geq 1$, for lin. ODE over I with continuous a_i we have:

1. Set $\mathcal{S}$ of $k\times$ diff. sol. $f : I \to \mathbb{C}(\mathbb{R})$ of the eq. is a complex (real) subspace of complex (real)-valued func. over I
2. $\dim(\mathcal{S}) = k\ \ \forall x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, exists unique $f \in \mathcal{S}$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)}(x_0) = y_{k-1}$. If a_i real-valued, same applies, but $\mathbb{C}$ replaced by $\mathbb{R}$.
3. Let b continuous on I. Exists solution f_0 to inhom. lin. ODE and $\mathcal{S}_b$ is set of funct. $f + f_0$ where $f \in \mathcal{S}$

The solution space $\mathcal{S}$ is spanned by k functions, which thus form a basis of $\mathcal{S}$. If inhomogeneous, $\mathcal{S}$ not vector space.

Finding solutions (in general)

(1) Find basis $\{f_1, \ldots, f_k\}$ for $\mathcal{S}_0$ for homogeneous equation (set $b(x) = 0$) (i.e. find homogeneous part, solve it)
(2) If inhomogeneous, find f_p that solves the equation. The set of solutions is then $\mathcal{S}_b = \{f_h + f_p \mid f_h \in \mathcal{S}_0\}$.
(3) If there are initial conditions, find equations $\in \mathcal{S}_b$ which fulfill conditions using SLE (as always)

2.3 Linear differential equations of first order

P 2.3.1: Solution of $y' + ay = 0$ is of form $f(x) = ze^{-A(x)}$ with A anti-derivative of a

Imhomogeneous equation

1. Plug all values into $y_p = \int b(x)e^{A(x)}$ ($A(x)$ in the exponent instead of $-A(x)$ as in the homogeneous solution)
2. Solve and the final $y(x) = y_h + y_p$. For initial value problem, determine coefficient z

2.4 Linear differential equations with constant coefficients

The coefficients a_i are constant functions of form $a_i(x) = k$ with k constant, where $b(x)$ can be any function.

Homogeneous Equation

1. Find *characteristic polynomial* (of form $\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_1\lambda + a_0$ for order k lin. ODE with coefficients $a_i \in \mathbb{R}$).
2. Find the roots of polynomial. The solution space is given by $\{z_j \cdot x^{v_j - 1}e^{\gamma_i x} \mid v_j \in \mathbb{N}, \gamma_i \in \mathbb{R}\}$ where v_j is the multiplicity of the root γ_i. For $\gamma_i = \alpha + \beta i \in \mathbb{C}$, we have $z_1 \cdot e^{\alpha x}\cos(\beta x)$, $z_2 \cdot e^{\alpha x}\sin(\beta x)$, representing the two complex conjugated solutions.

Inhomogeneous Equation

1. ***(Case 1)*** $b(x) = cx^d e^{\alpha x}$, with special cases x^d and $e^{\alpha x}$: $f_p = Q(x)e^{\alpha x}$ with Q a polynomial with $\deg(Q) \leq j + d$, where j is multiplicity of root α (if $P(\alpha) \neq 0$, then $j = 0$) of characteristic polynomial
2. ***(Case 2)*** $b(x) = cx^d\cos(\alpha x)$, or $b(x) = cx^d\sin(\alpha x)$: $f_p = Q_1(x) \cdot \cos(\alpha x) + Q_2(x9 \cdot \sin(\alpha x))$, where $Q_i(x)$ a polynomial with $\deg(Q_i) \leq d + j$, where j is the multiplicity of root αi (if $P(\alpha i) \neq 0$, then $j = 0$) of characteristic polynomial

Other methods

- ***Change of variable*** Apply substitution method here, substituting for example for $y' = f(ax + by + c)$ $u = ax + by$ to make the integral simpler. Mostly intuition-based (as is the case with integration by substitution)
- ***Separation of variables*** For equations of form $y' = a(y) \cdot b(x)$ (NOTE: Not linear), we transform into $\frac{y'}{a(y)} = b(x)$ and then integrate by substituting $y'(x)dx = dy$, changing the variable of integration. Solution: $A(y) = B(x) + c$, with $A = \int \frac{1}{a}$ and $B(x) = \int b(x)$. To get final solution, solve for the above equation for y.