1 Linear Algebra

Relevant definitions used throughout Analysis II.

$\mathbf{A} \in \mathbb{R}^{m\times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$

Def **Scalar Product** $x \cdot y := \sum_{i=0}^{n}(x_i \cdot y_i)$

Def **Euclidian Norm** $||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$

Used to generalize $|x|$ in many Analysis I definitions

Lem. **Properties of** $||x||$

(i) $||x|| \geq 0$
(ii) $||x|| \iff x = 0$
(iii) $||\alpha x|| = \alpha \cdot ||x||$
(iv) $||x+y|| \leq ||x|| + ||y||$ (Triangle Inequality)

Def **Definiteness**

Positive Definite $\overset{\text{def}}{\iff}$ $x^\top \mathbf{A} x > 0 \; \forall x \in \mathbb{R}^n_{\neq 0}$
Negative Definite $\overset{\text{def}}{\iff}$ $x^\top \mathbf{A} x < 0 \; \forall x \in \mathbb{R}^n_{\neq 0}$

If 0 is allowed, **A** is called positive/negative semi-definite.

Def **Trace** $\text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$

2 Differential Equations

Def **Differential Equation** (DE)
Equation relating unknown f to derivatives $f^{(i)}$ at *same* x.

Def **Ordinary Differential Equation** (ODE)
DE s.t. $f : I \to \mathbb{R}$ is in one variable.

Def **Partial Differential Equation** (PDE)
DE s.t. $f : I^d \to \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def **Order** $\text{ord}(F) := \max_{i\geq 0}\{i \mid f^{(i)} \in F, \; f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \to \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$\exists f : I \to \mathbb{R}$
s.t. $\forall x \in I : f'(x) = F(x, f(x))$ and $f(x_0) = y_0$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

2.1 Linear Differential Equations

Def **Linear Differential Equation** (LDE)

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

$I \subset \mathbb{R}$ is open, $\quad k \geq 1, \quad \forall i < k : a_i : I \to \mathbb{C}$

Def Homogeneity of LDEs
Homogeneous $\overset{\text{def}}{\iff}$ $b = 0$
Inhomogeneous $\overset{\text{def}}{\iff}$ $b \neq 0$

Remark $D(y) := y^{(k)} + \ldots + a_0 y$ is a linear operation:

$$D(z_1 f_1 + z_2 f_2) = z_1 D(f_1) + z_2 D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$ k-times differentiable

Def **Homogeneous Solution Space**
$\mathcal{S}(F) := \{f : I \to \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1 D(f_1) + z_2 D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$

Solutions for complex homogeneous LDEs

F s.t. $a_0, \; \ldots \;, a_{k-1}$ continuous and complex-valued
1. $\mathcal{S}$ is a complex vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. $a_0, \; \ldots \;, a_{k-1}$ continuous and real-valued
1. $\mathcal{S}$ is a real vector space, $\dim(\mathcal{S}) = k$
2. $\mathcal{S}$ is a subspace of $\{f \mid f : I \to \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def **Inhomogeneous Solution Space**
$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), \; f_0 \text{ is a particular sol.}\}$
Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. $a_0, \; \ldots \;, a_{k-1}$ continuous, $b : I \to \mathbb{C}$
1. $\forall x_0 \in I, (y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark **Applications of Linearity**
If f_1 solves F for b_1, and f_2 for b_2: $f_1 + f_2$ solves $b_1 + b_2$.
Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

3 Solutions to Differential Equations

3.1 Linear Solutions: First Order

Form: $y' + ay = b \qquad I \subset \mathbb{R}, \quad a, b : I \to \mathbb{R}$

Approach:

1. Hom. Solution f_1 for: $y' + ay = 0$
   Note that $\mathcal{S}$ has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for $\mathcal{S}$
2. Part. Solution f_0 for $y' + ay = b$

Solutions: $f_0 + zf_1 \quad$ for $z \in \mathbb{C}$

Explicit Homogeneous Solution

$A(x)$ is a primitive of a, $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$
$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

Method **Variation of Constants**: Treating z as $z(x)$ yields:

Explicit Inhomogeneous Solution

$A(x)$ is a primitive of a

$$f_0(x) = \underbrace{\left(\int b(x) \cdot \exp(A(x))\right)}_{z(x)} \cdot \exp(-A(x))$$

Method **Educated Guess**
Usually, y has a similar form to b:

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \cdot \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cdot \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$ae^{\alpha x} \cdot \sin(\beta x)$	$e^{\alpha x}(c \sin(\beta x) + d \cos(\beta x))$
$be^{\alpha x} \cdot \cos(\beta x)$	$e^{\alpha x}(c \sin(\beta x) + d \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x} \sin(\beta x)$	$e^{\alpha x}(R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x} \cos(\beta x)$	$e^{\alpha x}(R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

Remark If α, β are roots of $P(X)$ with multiplicity j, multiply guess with a $P_j(x)$.

3.2 Linear Solutions: Constant Coefficients

Form:

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$$

Where $a_0, \ldots, a_{k-1} \in \mathbb{C}$ are constants, $b(x)$ is continuous.

3.2.1 Homogeneous Equations

The idea is to find a Basis of $\mathcal{S}$:

Def **Characteristic Polynomial** $P(X) = \prod_{i=1}^{k}(X - \alpha_i)$

Remark The unique roots $\alpha_1, \ldots, \alpha_l$ form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}$$

$v_1, \ldots, v_k$ are the Multiplicities of $\alpha_1, \ldots, \alpha_k$

Remark If $\alpha_j = \beta + \gamma i \in \mathbb{C}$ is a root, $\bar{\alpha_j} = \beta - \gamma i$ is too. To get a real-valued solution, apply:

$$e^{\alpha_j x} = e^{\beta x}(\cos(\gamma x) + i \sin(\gamma x))$$

Explicit Homogeneous Solution

Using $\alpha_1, \ldots, \alpha_k$ from $P(X)$ s.t. $\alpha_i \neq \alpha_j$, $z_i \in \mathbb{C}$ arbitrary

$$f(x) = \prod_{i=1}^{k} z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)(x)} = \prod_{i=1}^{k} z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of $\mathcal{S}$

Solutions exist $\forall Z = (z_1, \ldots, z_k)$ since that system's $\det(M_Z) \neq 0$.

3.2.2 Inhomogeneous Equations

Method **Undetermined Coefficients**: An educated guess.

1. $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$
   $\deg(Q) \leq d + v_\alpha$, where v_α is α's multiplicity in $P(X)$
2. $\left.\begin{aligned} b(x) &= cx^d \cdot \cos(\alpha x) \\ b(x) &= cx^d \cdot \sin(\alpha x) \end{aligned}\right\} f_p = Q_1(x)\cos(\alpha x) + Q_2(x)\sin(\alpha x)$
   $\deg(Q_{1,2}) \leq d + v_\alpha$, where v_α is α's multiplicity in $P(X)$

Remark **Applying Linearity**
If $b(x) = \sum_{i=1}^{n} b_i(x)$, A solution for $b(x)$ is $f(x) = \sum_{i=1}^{n} f_i(x)$
Sometimes called *Superposition Principle* in this context

3.3 Other Methods

Method **Change of Variable**
If $f(x)$ is replaced by $h(y) = f(g(y))$, then h is a sol. too.
Changes like $h(t) = f(e^t)$ may lead to useful properties.

Separation of Variables

Form:

$$y' = a(y) \cdot b(x)$$

Solve using:

$$\int \frac{1}{a(y)}\, \mathrm{d}y = \int b(x)\, \mathrm{d}x + c$$

Usually $\int 1/a(y)\, \mathrm{d}y$ can be solved directly for $\ln|a(y)| + c$.

3.4 Method Overview

Method	Use case
Variation of constants	LDE with ord$(F) = 1$
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. $y' = a(y) \cdot b(x)$
Change of Variables	e.g. $y' = f(ax + by + c)$

4 Continuous functions in $\mathbb{R}^n$

Treating functions $f : X \subset \mathbb{R}^n \to \mathbb{R}/\mathbb{C}/\mathbb{R}^m, \quad m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \to \mathbb{R}^m$ means:
$x = (x_1, \ldots, x_n), \quad f(x) = f\big(f_1(x), \ldots, f_m(x)\big)$

4.1 Multivariate functions

Def **Linear map** $f : \mathbb{R}^n \to \mathbb{R}^m$
In other words: $f(x) = \mathbf{A}x, \quad \mathbf{A} \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

Def **Affine Linear map** $f(x) \mapsto \mathbf{A}x + c$

Def **Quadratic form** $Q : \mathbb{R}^n \to \mathbb{R}$
In other words: $Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} (a_{i,j} x_i x_j)$

Def **Monomials** $M(x) : \mathbb{R}^n \to \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$
For example: $f(x, y, z) = 16x^2yz^5$

Def $\deg(M) := e = \sum_{i=1}^{n} d_i$
For example: $\deg(16x^2yz^5) = 8$

Def **Polynomials** $P(x) := \sum_{i=0}^{n} M_i(x)$
For example: $P(x, y, z) = x^3 + 25x^2y^6z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$
For example: $\deg(x^3 + 25x^2y^6z + xy) = 9$

Visualisations for some function types:

Def **Graph** $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$
Only for $f : \mathbb{R}^2 \to \mathbb{R}$. Visually, this is a surface in $\mathbb{R}^3$

Def **Vector Plots** for $f : \mathbb{R}^2 \to \mathbb{R}^2$
Points in $(x, y) \in \mathbb{R}^2$ are displayed as vectors $f(x, y)$

4.2 Sequences in $\mathbb{R}^n$

Def **Sequences in** $\mathbb{R}^n$
$(x_k)_{k \geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \ldots x_{k,n})$

Def **Convergence in** $\mathbb{R}^n$

$$\lim_{k\to\infty} \Big(x_k\Big) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \quad ||x_k - y|| < \epsilon$$

Using this definition preserves many familiar results:

Lem. **Equivalent conditions to Convergence**

$(i) \quad \forall i$ s.t. $1 \leq i \leq n : \quad \lim_{k\to\infty} \Big(x_{k,i}\Big) = y_i$

$(ii) \quad \lim_{k\to\infty} \Big\|x_k - y\Big\| = 0$

Def **Limits at points**

$$\lim_{x \neq x_0 \to x_0} \Big(f(x)\Big) = y \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$$
$$\forall x \neq x_0 \in X : \big\|x - x_0\big\| < \delta \implies \big\|f(x) - y\big\| < \epsilon$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m, \quad x_0 \in X, \quad y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

4.3 Continuity in $\mathbb{R}^n$

Def **Continuity in** $\mathbb{R}^n$
f continuous at $x_0 \in X \overset{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\big\|x - x_0\big\| < \delta \implies \big\|f(x) - f(x_0)\big\| < \epsilon$$

f continuous $\overset{\text{def}}{\iff} \forall x \in X : f$ continuous at x
$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. **Continuitiy using Sequences**
f continuous at x_0 if and only if:

$$\forall (x_k)_{k\geq 1} : \quad \lim_{k\to\infty} \Big(x_k\Big) = x_0 \implies \lim_{k\to\infty} \Big(f(x_k)\Big) = f(x_0)$$

$X \subset \mathbb{R}^n, \quad f : X \to \mathbb{R}^m$

Lem. **Continuity of Compositions**
$f : X \to Y, \ g : Y \to \mathbb{R}^p$ continuous $\implies g \circ f$ continuous
$X \subset \mathbb{R}^n, \quad Y \subset \mathbb{R}^m, \quad p \geq 1$

Lem. **Continuity using Coordinate Functions**
$f : \mathbb{R}^n \to \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

4.4 Subsets of $\mathbb{R}^n$

Def **Bounded**
$X \subset \mathbb{R}^n$ bounded $\overset{\text{def}}{\iff} \Big\{\|x\| \mid x \in X\Big\} \subset \mathbb{R}$ bounded.
Example: The open disc $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is bounded.

Def **Closed**
$X \subset \mathbb{R}^n$ closed $\overset{\text{def}}{\iff} \forall (x_k)_{k\geq 1} \in X : \quad \lim_{x\to\infty} \Big(x_k\Big) \in X$
Example: $\emptyset, \mathbb{R}^n$ are closed.

Def **Compact** if closed and bounded.
Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is compact.

Def **Open**
$X \subset \mathbb{R}^n$ open $\overset{\text{def}}{\iff} \forall x \in X, \ \exists \delta > 0 :$

$$\Big\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \ \ \forall i \leq n\Big\} \subset X$$

In other words: Changing any coord. x_i by δ keeps x' in X
Example: $\emptyset, \mathbb{R}^n$ are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. **Continous functions preserve closed/open**

$\forall$ closed/open Y :

$$f^{-1}(Y) = \Big\{x \in \mathbb{R}^n \mid f(x) \in Y\Big\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $\quad Y \subset \mathbb{R}^m$

Lem. **The complement of open sets is closed**

$$X \subset \mathbb{R}^n \text{ is open} \iff \underbrace{\Big\{x \in \mathbb{R}^n \mid x \notin X\Big\}}_{\text{Complement}} \text{ is closed}$$

Min-Max Theorem

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \to \mathbb{R}$:

$$\exists x_1, x_2 \in X : \quad f(x_1) = \sup_{x\in X} f(x), \quad f(x_2) = \inf_{x\in X} f(x)$$

5 Differential Calculus in $\mathbb{R}^n$

5.1 Partial Derivatives

Partial Derivative

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$, $1 \le i \le n$, $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \to \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,i}, \dots, x_{0,t-1},\ t\ , x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \to t}$$

Notation $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

Lem. Properties of Partial Derivatives

Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

(i) $\partial x_i(f+g) = \partial x_i f + \partial x_i g$

(ii) $\partial x_i(fg) = \partial x_i(f)g + \partial x_i(g)f$ if $m = 1$

(iii) $\partial x_i\left(\frac{f}{g}\right) = \frac{\partial x_i(f)g - \partial x_i(g)f}{g^2}$ if $g(x) \neq 0\ \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f.g : X \to \mathbb{R}^n$, $1 \le i \le n$

The Jacobian

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial x_1 f_1(x) & \partial x_2 f_1(x) & \cdots & \partial x_n f_1(x) \\ \partial x_1 f_2(x) & \partial x_2 f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_1 f_n(x) & \partial x_2 f_n(x) & \cdots & \partial x_n f_m(x) \end{bmatrix}$$

Think of f as a vector of f_i, then $\mathbf{J}_f$ is that vector stretched for all x_j

Def Gradient $\nabla f(x_0) := \begin{bmatrix} \partial x_1 f(x_0) \\ \vdots \\ \partial x_n f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$, i.e. *must* map to 1 dimension

Remark ∇f points in the direction of greatest increase.

This generalizes that in $\mathbb{R}$, sgn(f) shows if f increases/decreases

Def Divergence div$(f)(x_0) :=$ Tr$\big(\mathbf{J}_f(x_0)\big)$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$, $\mathbf{J}_f$ exists

5.2 The Differential

Partial derivatives don't provide a good approx. of f, unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in $\mathbb{R}^n$.

Differentiability in $\mathbb{R}^n$ & the Differential

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If f is differentiable at $x_0 \in X$ with u s.t.

$$\lim_{x \neq x_0 \to x_0} \frac{1}{\|x - x_0\|}\Big(f(x) - f(x_0) - u(x - x_0)\Big) = 0$$

Similarly, f is differentiable if this holds for all $x \in X$

Lem. Properties of Differentiable Functions

(i) Continuous on X

(ii) $\forall i \le m, j \le n :$ $\partial_{x_j} f_i$ exists

(iii) $m = 1 :$ $\partial_{x_i} f(x_0) = a_i$

for: $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ differentiable on X

Lem. Preservation of Differentiability

(i) $f + g$ is differentiable: $d(f+g) = df + dg$

(ii) fg is differentiable, if $m = 1$

(iii) $\dfrac{f}{g}$ is differentiable, if $m = 1$, $g(x) \neq 0\ \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f, g : X \to \mathbb{R}^m$ differentiable on X

Lem. Cont. Partial Derivatives imply Differentiability

if all $\partial_{x_j} f_i$ exist and are continuous:

$$f \text{ differentiable on } X, \quad df(x_0) = \mathbf{J}_f(x_0)$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$

Lem. Chain Rule $g \circ f$ is differentiable on X

$$\begin{aligned} d(g \circ f)(x_0) &= dg\big(f(x_0)\big) \circ df(x_0) \\ \mathbf{J}_{g \circ f}(x_0) &= \mathbf{J}_g\big(f(x_0)\big) \cdot \mathbf{J}_f(x_0) \end{aligned}$$

$X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open, $f : X \to Y, g : Y \to \mathbb{R}^p, f, g$ diff.-able

Def Tangent Space

$$T_f(x_0) := \Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\Big\}$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ diff.-able, $x_0 \in X$, $u = df(x_0)$

Def Directional Derivative

$$D_v f(x_0) = \lim_{t \neq 0 \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$, $v \neq 0 \in \mathbb{R}^n$, $x_0 \in X$

Lem. Directional Derivatives for Diff.-able Functions

$$D_v f(x_0) = df(x_0)(v) = \mathbf{J}_f(x_0) \cdot v$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ diff.-able, $v \neq 0 \in \mathbb{R}^n$, $x_0 \in X$

Remark $D_v f$ is linear w.r.t v, so: $D_{v_1 + v_2} f = D_{v_1} f + D_{v_2} f$

Remark $D_v f(x_0) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$

In the case $f : X \to \mathbb{R}$, where θ is the angle between v and $\nabla f(x_0)$

5.3 Higher Derivatives

Def **Differentiability Classes**

$$f \in C^1(X;\mathbb{R}^m) \overset{\text{def}}{\iff} f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \text{ exist}$$
$$f \in C^k(X;\mathbb{R}^m) \overset{\text{def}}{\iff} f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \in C^{k-1}$$
$$f \in C^\infty(X;\mathbb{R}^m) \overset{\text{def}}{\iff} f \in C^k(X;\mathbb{R}^m)\ \forall k \geq 1$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$

Lem. Polynomials, Trig. functions and $\exp$ are in C^∞

Lem. **Operations preserve Differentiability Classes**

$$\begin{array}{lll} (i) & f+g & \in C^k \\ (ii) & fg & \in C^k \quad \text{if } m=1 \\ (iii) & \dfrac{f}{g} & \in C^k \quad \text{if } m=1, g(x)\neq 0\ \forall x \in X \end{array}$$

$f, g \in C^k$

Lem. **Composition preserves Differentiability Classes**

$$g \circ f \in C^k$$

$f \in C^k$, $f(X) \subset Y$, $Y \subset \mathbb{R}^m$ open, $g : Y \to \mathbb{R}^p$, $g \in C^k$

Partial Derivatives commute in C^k

$k \geq 2$, $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$, $f \in C^k$

$$\forall x, y : \quad \partial_{x,y} f = \partial_{y,x} f$$

This generalizes for $\partial_{x_1,\ldots,x_n} f$.

Remark Linearity of Partial Derivatives

$$\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$$

Assuming both $\partial_x f_{1,2}$ exist.

Def **Laplace Operator**

$$\Delta f := \text{div}\big(\nabla f(x)\big) = \sum_{i=0}^n \frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_i}\Big) = \sum_{i=0}^n \frac{\partial^2 f}{\partial x_i^2}$$

The Hessian

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$, $f \in C^2$, $x_0 \in X$

$$\mathbf{H}_f(x) := \begin{bmatrix} \partial_{1,1} f(x_0) & \partial_{2,1} f(x_0) & \cdots & \partial_{n,1} f(x_0) \\ \partial_{1,2} f(x_0) & \partial_{2,2} f(x_0) & \cdots & \partial_{n,2} f(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1,n} f(x_0) & \partial_{2,n} f(x_0) & \cdots & \partial_{n,n} f(x_0) \end{bmatrix}$$

Where $\big(\mathbf{H}_f(x)\big)_{i,j} = \partial_{x_i,x_j} f(x)$

Note that $f : X \to \mathbb{R}$, i.e. $\mathbf{H}_f$ only exists for 1-dimensionally valued f

Notation $\mathbf{H}_f(x) = \text{Hess}_f(x) = \nabla^2 f(x)$

Remark $\mathbf{H}_f(x_0)$ is symmetric: $\big(\mathbf{H}_f(x_0)\big)_{i,j} = \big(\mathbf{H}_f(x_0)\big)_{j,i}$

Def **Polar Coordinates**

$$g(r,\theta) = \big(r\cos(\theta), r\sin(\theta)\big)$$
$$\mathbf{J}_g(r,\theta) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$
$$\partial_x f = \cos(\theta)\partial_r f - \frac{1}{r}\sin(\theta)\partial_\theta f$$
$$\partial_y f = \sin(\theta)\partial_r f + \frac{1}{r}\cos(\theta)\partial_\theta f$$

$(r,\theta) \in (0,+\infty) \times \mathbb{R}$, $\det(\mathbf{J}_g) = r$

5.4 Taylor Polynomials

Def $|m| := \sum_{i=1}^n m_1$

Def $y^m := y_1^m \cdots y_n^m$

Def $m! := m_1! \cdots m_n!$

for $m = (m_1, \ldots, m_n)$, $y = (y_1, \ldots, y_n)$

Taylor Polynomials

$k \geq 1$, $f : X \to \mathbb{R}$, $f \in C^k$, $x_0 \in X$

$$T_k f(y; x_0) := \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

Lem. **Taylor Approximation**

$$\lim_{x \neq x_0 \to x_0} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

Where $f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$

$k \geq 1$, $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$, $f \in C^k$, $x_0 \in X$

Remark Taylor polynomials of degree $1, 2$:

$$T_1 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y$$
$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2}\Big(x_0^\top \cdot \mathbf{H}_f(y) \cdot x_0\Big)$$

Method Calculating $T_k f(y; x_0)$ also yields $\mathbf{H}_f$ for $k \geq 2$.

$$T_2 f((x_0, y_0); (x, y)) = \ldots + ax^2 + by^2 + cxy$$
$$\implies \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} 2a & c \\ c & 2b \end{bmatrix}$$

Method Taylor Polynomials can be found by combination.

Example: $f(x,y) = \underbrace{e^{y^4}}_{1} + \underbrace{\sin(xy)}_{2} + \underbrace{2xy^2}_{3} - \underbrace{\ln(x^2+1)}_{4}$, $k = 3$

1. $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \implies e^{y^4} \approx 1 + y^4 + \frac{y^8}{2} + \frac{y^12}{6}$
   Since $k = 3$, discarding all terms with $\deg > 3$ yields: $e^{y^3} \approx 1$
2. $\sin(x) \approx x - \frac{x^3}{6} \implies \sin(xy) \approx xy$
3. $2xy^2 \approx 2xy^2$ (Since it's already a polynomial, $\deg = 3$)
4. $\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \implies \ln(x^2+1) \approx x^2$

Thus: $f(x) \approx 1 + xy + 2xy^2 - x^2 = T_3 f\Big((0,0); (x,y)\Big)$

5.5 Critical Points

Lem. Local Maxima & Minima

$$\left.\begin{array}{l} f(y) \leq f(x_0)\ \forall y \text{ close} \\ f(y) \geq f(x_0)\ \forall y \text{ close} \end{array}\right\} \quad \frac{\partial f}{\partial x_i}(x_0) = 0 \quad \forall i \leq n$$

In other words: $df(x_0) = \nabla f(x_0) = 0$

$f : X \to \mathbb{R}, \quad X \subset \mathbb{R}^n$ open, f diff.-able

Def Critical Point

$$x_0 \in X \text{ is critical} \quad \overset{\text{def}}{\iff} \quad \nabla f(x_0) = 0$$

$X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$ diff.-able

Remark Existance of Maxima/Minima

Don't *have to* exist if X is open, only if X is compact.

However, for compact sets, the lemma above no longer applies.

Method Critical points on Compact Sets

Decompose $X = X' \cup B$, s.t. X' is open, B is a *boundary*.

1. Find critical points in X'
2. Check if any $x \in B$ is a maximum/minimum

Def Non-degenerate Critical Point

$$x_0 \in X \text{ non-deg.} \quad \overset{\text{def}}{\iff} \quad \det\Big(\mathbf{H}_f(x_0)\Big) \neq 0$$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$ is critical

Lem. Definiteness of the Hessian

$\mathbf{H}_f(x_0)$ positive definite $\implies x_0$ is a local min.

$\mathbf{H}_f(x_0)$ negative definite $\implies x_0$ is a local max.

$\mathbf{H}_f(x_0)$ indefinite $\implies x_0$ is a saddle point.

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$ non-deg. critical

Method Determining Definiteness for 2×2 Matrices

$$\begin{tikzcd}
\text{p. semi-def.} & & & & \text{pos. def.} \\
& \mathrm{Tr}(A) \arrow[ul, "\text{pos.}"] \arrow[dl, "\text{neg.}"] \arrow[d, "0"] & \det(A) \arrow[l, "0"'] \arrow[r, "\text{pos.}"] \arrow[d, "\text{neg.}"] & \mathrm{Tr}(A) \arrow[ur, "\text{pos.}"] \arrow[dr, "\text{neg.}"] & \\
\text{n. semi-def.} & A \text{ is zero} & \text{indefinite} & & \text{neg. def.}
\end{tikzcd}$$

6 Integral Calculus in $\mathbb{R}^n$

6.1 Line Integrals

Integrals for $f : I \to \mathbb{R}^n$

$I = [a,b]$ closed & bounded, $\quad f : I \to \mathbb{R}^n$ cont.

$$\int_a^b f(t)\ dt = \left(\int_a^b f_1(t)\ dt, \ldots, \int_a^b f_n(t)\ dt\right)$$

Def Piecewise Continuity

$\exists k \geq 1$, and a Partition $a = t_0 < \cdots < t_k = b$

s.t. $f_j : [t_{j-1}, t_j] \to \mathbb{R}^n$ has $f_j \in C^1$ for all $j \leq k$

For $f : I \to \mathbb{R}^n$

Def Parametrized Curve $\gamma : [a,b] \to \mathbb{R}^n$ pw.-cont.

Also called *Path* from $\gamma(a)$ to $\gamma(b)$

Line Integral

$\gamma : [a,b] \to \mathbb{R}^n$ is path, $\quad X \subset \mathbb{R}^n$ s.t. $\gamma([a,b]) \subset X$

$f : X \to \mathbb{R}^n$ continuous

$$\int_\gamma f(s) \cdot\ ds := \int_a^b f\Big(\gamma(t)\Big) \cdot \gamma'(t)\ dt$$

Def Continuous integrals are linear

$$\int_a^b \Big(f(t) + g(t)\Big)\ dt = \int_a^b f(t)\ dt + \int_a^b g(t)\ dt$$

$f, g : I \to \mathbb{R}^n$ continuous

Remark $f : X \to \mathbb{R}^n$ is called a *Vector Field*.

Def Oriented Reparametrization

For $\gamma : [a,b] \to \mathbb{R}^n$ (param. curve), $\phi : [c,d] \to [a,b]$ continuous

$$\sigma : [c,d] \to \mathbb{R}^n \text{ s.t. } \sigma = \gamma \circ \phi$$

diff.-able on (c,d), strictly increasing and $\phi(c) = a, \phi(d) = b$

Lem. Oriented Reparametrizations preserve Integrals

$$\int_\gamma f(s) \cdot ds = \int_\sigma f(s) \cdot ds$$

$\gamma : [a,b] \to \mathbb{R}^n$ param. curve, $\quad \sigma$ oriented reparam.,

$\gamma([a,b]) \subset X, \quad f : X \to \mathbb{R}^n$ cont.

Remark Line Integrals of the form $\int_\gamma \nabla f(s) \cdot ds$ have:

$$\int_\gamma \nabla f(s) \cdot ds = \int_a^b \sum_{i=1}^n \frac{\partial g}{\partial x_i}\Big(\gamma(t)\Big)\gamma_i'(t) = f\Big(\gamma(b)\Big) - f\Big(\gamma(a)\Big)$$

Follows from the Chain rule for $h(t) = g(\gamma(t))$

$X \subset \mathbb{R}^n$ open, $\quad f : X \to \mathbb{R}, \quad f \in C^1, \quad \gamma : [a,b] \to X$ param. curve

Def Conservative Vector Field

$f : X \to \mathbb{R}$ conservative $\overset{\text{def}}{\iff} \forall \gamma_1, \gamma_2$ s.t. start & end points match:

$$\int_{\gamma_1} f(s) \cdot ds = \int_{\gamma_2} f(s) \cdot ds$$

No matter which path, if start & end match, the integral matches

Remark Closed Curves in Conservative Vector Fields

$$\forall\ \gamma : [a,a] \to \mathbb{R} : \quad \int_\gamma f(s) \cdot ds = 0$$

This is actually equivalent to f being conservative.

The Potential exists in Conservative Vector Fields

$X \subset \mathbb{R}^n$ open, $\quad f$ conservative

$$\exists g \in C^1 : \quad f = \nabla g$$

If $x_1, x_2 \in X$ are joined by a γ, g is unique up to $C \in \mathbb{R}$

$$\nabla g_1 = f \implies g - g_1 \text{ is constant on } X$$

Def Path-Connected Set

$\forall x_1, x_2 \in X : \exists \gamma : [a,b] \to X$ s.t. $\gamma(a) = x_1, \gamma(b) = x_2$

Lem. Property of Conservative Vector Fields

Easy way to e.g. disprove f being conservative:

$$\forall 1 \leq i \neq j \leq n: \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

$X \subset \mathbb{R}^n$ open, $\quad f: X \to \mathbb{R}^n, \quad f \in C^1, \quad f$ conserv.

Only this was: This being true does not imply f is conservative!

Def Star Shaped Set

$\exists x_0 \in X : \forall x \in X$ Line seg. $x_0 \to x$ is in X

Def Convex Set

$\forall x_1, x_2 \in X$: Line seg. $x_1 \to x_2$ is in X

Convex implies star shaped.

Th. Some Star Shaped Sets are conservative

In open star-shaped sets $X \subset \mathbb{R}^n$: $\quad f \in C^1$

$$\forall 1 \leq i \neq j \leq n: \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \implies f \text{ conservative}$$

Def $\mathrm{curl}(f) := \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix}$ $\quad f: X \to \mathbb{R}^3, \quad f \in C^1$

Remark $\mathrm{curl}(f) = 0 \iff \forall 1 \leq i \neq j \leq 3 : \dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i}$

6.2 The Riemann Integral in $\mathbb{R}^n$

For $f : X \to \mathbb{R}$ ($X \subset \mathbb{R}^n$ bounded & closed), $\int_X f(x)\, dx$ fulfills:

1. **Composability**
   $\int_X f(x)\, dx = \int_a^b f(x)\, dx \quad n = 1, X = [a, b]$
2. **Linearity**
   $\int_X \Big(af_1(x) + bf_2(x)\Big)\, dx = a \int_X f_1(x)\, dx + b \int_X f_2(x)\, dx$
   f, g cont. on X, $a, b \in \mathbb{R}$
3. **Positivity**
   $f \leq g \implies \int_X f(x)\, dx \leq \int_X g(x)\, dx$
4. **Upper Bound**
   $\left|\int_X f(x)\, dx\right| \leq \int_X |f(x)|\, dx$
5. **Triangle Inequality**
   $\left|\int_X \Big(f(x) + g(x)\Big)\, dx\right| \leq \int_X |f(x)|\, dx + \int_X |g(x)|\, dx$
6. **Volume**
   $\int_X f(x)\, dx$ is the volume of $\Big\{(x, y) \in X \times \mathbb{R} \;\Big|\; 0 \leq y \leq f(x)\Big\}$
   So the intuitive idea of $\int_a^b f(x)\, dx$ being the area carries over.
7. **Domain Additivity**
   $\int_{X_1 \cup X_2} f(x)\, dx + \int_{X_1 \cap X_2} f(x)\, dx = \int_{X_1} f(x)\, dx + \int_{X_2} f(x)\, dx$
   If X_1, X_2 are compact, f is cont. on $X_1 \cup X_2$

Fubini's Theorem: Multiple Integrals

$f : X \to \mathbb{R}, \quad n = n_1 + n_2, \quad n_1, n_2 \geq 1$

$$\begin{aligned} X_{x_1} &:= \Big\{x_2 \in \mathbb{R}^{n_2} \;\Big|\; (x_1, x_2) \in X\Big\} \subset \mathbb{R}^{n_2} \\ X_1 &:= \Big\{x_1 \in \mathbb{R}^{n_1} \;\Big|\; X_{x_1} \neq \emptyset\Big\} \subset \mathbb{R}^{n_1} \end{aligned}$$

If $g(x_1) := \int_{X_{x_1}} f\Big((x_1, x_2)\Big)\, dx_2$ is continuous on X_1:

$$\int_X f(x)\, dx = \int_{X_1} \left(\int_{X_{x_1}} f\Big((x_1, x_2)\Big)\, dx_2\right) dx_1$$

The role of x_1, x_2 can be swapped, if f is continuous.