Analysis II

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TITLE PAGE COMING SOON

"*Multiply it by ai*"
- Özlem Imamoglu, 2025

HS2025, ETHZ

Cheat-Sheet based on Lecture notes and Script

https://metaphor.ethz.ch/x/2025/hs/401-0213-16L/sc/script-analysis-II.pdf

Contents

1 Introduction

This Cheat-Sheet does not serve as a replacement for solving exercises and getting familiar with the content. There is no guarantee that the content is 100% accurate, so use at your own risk. If you discover any errors, please open an issue or fix the issue yourself and then open a Pull Request here:

`https://github.com/janishutz/eth-summaries`

This Cheat-Sheet was designed with the HS2025 page limit of 10 A4 pages in mind. Thus, the whole Cheat-Sheet can be printed full-sized, if you exclude the title page, contents and this page. You could also print it as two A5 pages per A4 page and also print the Analysis I summary in the same manner, allowing you to bring both to the exam.

And yes, she did really miss an opportunity there with the quote... But she was also sick, so it's not as unexpected

This summary also uses tips and tricks from this Exercise Session

2 Differential Equations

2.1 Introduction

Ex 2.1.1: $f'(x) = f(x)$ has only solution $f(x) = ae^x$ for any $a \in \mathbb{R}$; $f' - a = 0$ has only solution $f(x) = \int_{x_0}^{x} a(t)\ \mathrm{d}t$

T 2.1.2: Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a differential function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. The Ordinary Differential Equation (ODE) $y' = F(x, y)$ has a unique solution f defined on a "largest" interval I that contains x_0 such that $y_0 = f(x_0)$

2.2 Linear Differential Equations

An ODE is considered linear if and only if the ys are only scaled and not part of powers.
D 2.2.1: *(Linear differential equation of order k)* (order = highest derivative) $y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b$, with a_i and b functions in x. If $b(x) = 0\ \ \forall x$, ***homogeneous***, else ***inhomogeneous***
T 2.2.2: For open $I \subseteq \mathbb{R}$ and $k \geq 1$, for lin. ODE over I with continuous a_i we have:

1. Set $\mathcal{S}$ of $k\times$ diff. sol. $f : I \to \mathbb{C}(\mathbb{R})$ of the eq. is a complex (real) subspace of complex (real)-valued func. over I
2. $\dim(\mathcal{S}) = k\ \ \forall x_0 \in I$ and any $(y_0, \ldots, y_{k-1}) \in \mathbb{C}^k$, exists unique $f \in \mathcal{S}$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, \ldots, f^{(k-1)}(x_0) = y_{k-1}$. If a_i real-valued, same applies, but $\mathbb{C}$ replaced by $\mathbb{R}$.
3. Let b continuous on I. Exists solution f_0 to inhom. lin. ODE and $\mathcal{S}_b$ is set of funct. $f + f_0$ where $f \in \mathcal{S}$

The solution space $\mathcal{S}$ is spanned by k functions, which thus form a basis of $\mathcal{S}$. If inhomogeneous, $\mathcal{S}$ not vector space.

Finding solutions (in general)

(1) Find basis $\{f_1, \ldots, f_k\}$ for $\mathcal{S}_0$ for homogeneous equation (set $b(x) = 0$) (i.e. find homogeneous part, solve it)
(2) If inhomogeneous, find f_p that solves the equation. The set of solutions is then $\mathcal{S}_b = \{f_h + f_p \mid f_h \in \mathcal{S}_0\}$.
(3) If there are initial conditions, find equations $\in \mathcal{S}_b$ which fulfill conditions using SLE (as always)

2.3 Linear differential equations of first order

P 2.3.1: Solution of $y' + ay = 0$ is of form $f(x) = Ce^{-A(x)}$ with A anti-derivative of a

Inhomogeneous equation

1. Plug all values into $y_p = \int b(x)e^{A(x)}$ ($A(x)$ in the exponent instead of $-A(x)$ as in the homogeneous solution)
2. Solve and the final $y(x) = y_h + y_p$. For initial value problem, determine coefficient C

2.4 Linear differential equations with constant coefficients

The coefficients a_i are constant functions of form $a_i(x) = k$ with k constant, where $b(x)$ can be any function.

Homogeneous Equation

1. Find ***characteristic polynomial*** (of form $\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_1\lambda + a_0$ for order k lin. ODE with coefficients $a_i \in \mathbb{R}$).
2. Find the roots of polynomial. The solution space is given by $\{C_j \cdot x^{v_j - 1}e^{\gamma_i x} \mid v_j \in \mathbb{N}, \gamma_i \in \mathbb{R}\}$ where v_j is the multiplicity of the root γ_i and C_j is a constant. For $\gamma_i = \alpha + \beta i \in \mathbb{C}$, we have $C_1 \cdot e^{\alpha x}\cos(\beta x)$, $C_2 \cdot e^{\alpha x}\sin(\beta x)$, representing the two complex conjugated solutions.

The homogeneous equation will then be all the elements of the set summed up.

Inhomogeneous Equation

1. ***(Case 1)*** $b(x) = cx^d e^{\alpha x}$, with special cases x^d and $e^{\alpha x}$: $f_p = Q(x)e^{\alpha x}$ with Q a polynomial with $\deg(Q) \leq j + d$, where j is multiplicity of root α (if $P(\alpha) \neq 0$, then $j = 0$) of characteristic polynomial
2. ***(Case 2)*** $b(x) = cx^d\cos(\alpha x)$, or $b(x) = cx^d\sin(\alpha x)$: $f_p = Q_1(x) \cdot \cos(\alpha x) + Q_2(x9 \cdot \sin(\alpha x))$, where $Q_i(x)$ a polynomial with $\deg(Q_i) \leq d + j$, where j is the multiplicity of root αi (if $P(\alpha i) \neq 0$, then $j = 0$) of characteristic polynomial

Other methods

- ***Change of variable*** Apply substitution method here, substituting for example for $y' = f(ax + by + c)$ $u = ax + by$ to make the integral simpler. Mostly intuition-based (as is the case with integration by substitution)
- ***Separation of variables*** For equations of form $y' = a(y) \cdot b(x)$ (NOTE: Not linear), we transform into $\frac{y'}{a(y)} = b(x)$ and then integrate by substituting $y'(x)dx = dy$, changing the variable of integration. Solution: $A(y) = B(x) + c$, with $A = \int \frac{1}{a}$ and $B(x) = \int b(x)$. To get final solution, solve for the above equation for y.

3 Differential Calculus in Vector Space

3.2 Continuity

D 3.2.1: *(Convergence in $\mathbb{R}^n$)* Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$ with $x_k = (x_{k,1},\dots,x_{k,n})$ and let $y = (y_1,\dots,y_n) \in \mathbb{R}^n$. (x_k) converges to y as $k \to +\infty$ if $\forall\varepsilon > 0 \;\; \exists N \geq 1$ s.t. $\forall n \geq N$ we have $||x_k - y|| < \varepsilon$

L 3.2.2: (x_k) converges to y as $k \to +\infty$ iff one of following equiv. statements holds: ***(1)*** $\forall 1 \leq i \leq n$, the sequence $(x_{k,i})$ with $x_{k,i} \in \mathbb{R}$ converges to y_i ***(2)*** $(||x_k - y||)$ converges to 0 as $k \to +\infty$

D 3.2.3: *(Continuity)* Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. ***(1)*** Let $x_0 \in X$. f continuous in $\mathbb{R}^n$ if $\forall\varepsilon > 0 \;\; \exists\delta > 0$ s.t. if $x \in X$ satisfies $||x - x_0|| < \delta$, then $||f(x) - f(x_0)|| < \varepsilon$ ***(2)*** f continuous *on* X if continuous at $x_0 \;\; \forall x_0 \in X$ **P 3.2.4:** Let X and f as prev. Let $x_0 \in X$. f continuous at x_0 iff $\forall (x_k)_{k\geq 1}$ in X s.t. $x_k \to x_0$ as $k \to +\infty$, $(f(x_k))_{k\geq 1}$ in $\mathbb{R}^m$ converges to $f(x)$

D 3.2.5: *(Limit)* Let X, f and x_0 as prev. and $y \in \mathbb{R}^m$. f *has limit* y as $x \to x_0$ with $x \neq x_0$ if $\forall\varepsilon > 0 \;\; \exists\delta > 0$ s.t. $\forall x \neq x_0 \in X, ||x - x_0|| < \delta$ we have $||f(x) - y|| < \varepsilon$. We write $\lim_{\substack{x\to x_0\\x\neq x_0}} f(x) = y$ **R 3.2.6:** Also possible without ass. that $x_0 \in X$

P 3.2.7: Let X, f, x_0 and y as prev. We have $\lim_{\substack{x\to x_0\\x\neq x_0}} f(x) = y$ iff $\forall (x_k)$ in X s.t. $x_k \to x$ as $k \to +\infty$ and $x_k \neq x_0$ $(f(x_k))$ in $\mathbb{R}^m$ converges to y **P 3.2.9:** Let $X \subseteq \mathbb{R}^n$, $y \subseteq \mathbb{R}^m$, $p \in \mathbb{N}$ and let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be cont. Then $g \circ f$ is continuous

Remark: To find the limits, we have two tricks (for $\lim\limits_{(x,y)\to(a,b)}$):

1. ***(Substitution)*** Substitute $y = x + (b - a)$, then limit is $\lim\limits_{x\to(a-b)}$
2. ***(Polar coordinates)*** Substitute $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ and the limit is $\lim\limits_{r\to 0}$

Ex 3.2.10: ***(1)*** $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ continuous $\Rightarrow f = (f_1, f_2) : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2}$ is continuous (Cartesian product) ***(2)*** Any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. In particular, the identity map is continuous ***(3)*** If $f_1,\dots,f_n$ continuous, then $f(x_1,\dots,x_n) = f_1(x_1)\cdot\ldots\cdot f_n(x_n)$ is continuous ***(4)*** Polynomials in $x_1,\dots,x_n$ are continuous ***(5)*** f_1f_2 is continuous if f_1 and f_2 are continuous and if $f_2(x) \neq 0 \;\; \forall x \in X$, then $f_1 \div f_2$ is continuous. (see Theorem 2.1.8 in Analysis I)
(6) If both f and g have limits, then $\lim\limits_{x\to x_0}(f(x) + g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$ and analogous for $\times$ ***(7)*** If $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, then $g(x) = f(x, y_0)$ for $y_0 \in \mathbb{R}$ is continuous. The converse is not true

D 3.2.11: ***(1)*** $X \subseteq \mathbb{R}^n$ is ***bounded*** if the set of $||x||$ for $x \in X$ is bounded in $\mathbb{R}$ ***(2)*** $X \subseteq \mathbb{R}^n$ is ***closed*** if $\forall (x_k)$ in X that converge in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$ ***(3)*** $X \subseteq \mathbb{R}^n$ is ***compact*** if it is bounded and closed

Ex 3.2.12: ***(1)*** $\emptyset$ and $\mathbb{R}^n$ are closed. ***(2)*** The *open* disc $D = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$ for $r > 0$ and $x_0 \in \mathbb{R}^n$ is bounded and not closed. ***(3)*** The *closed* disc $\Delta = \{x \in \mathbb{R}^n : ||x - x_0|| \leq r\}$ is bounded and closed. In particular, a closed interval is a closed set. An interval is compact if it is bounded ***(4)*** If $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^m$ are bounded (also closed or compact), then so is $X_1 \times X_2 \subseteq \mathbb{R}^{n+m}$

P 3.2.13: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed $Y \subseteq \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subseteq \mathbb{R}^n$ is closed

Ex 3.2.14: The ***zero set*** $Z = \{x \in \mathbb{R}^n : f(x) = 0\}$ is closed in $\mathbb{R}^n$ because $\{0\} \subseteq \mathbb{R}$ is closed. More generally: for any $r \geq 0$, $\{x \in \mathbb{R}^n : |f(x)| \leq r\}$ is $f^{-1}([-r, r])$ and is closed, since $[-r, r]$ is closed. Furthermore: $\{x \in \mathbb{R}^3 : ||x - x_0|| = r\}$ is closed

T 3.2.15: Let $(X \neq \emptyset) \subseteq \mathbb{R}^n$ compact and $f : X \to \mathbb{R}$ continuous. Then f bounded, has max and min, i.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup\limits_{x\in X} f(x)$ and $f(x_-) = \inf\limits_{x\in X} f(x)$

3.3 Partial derivatives

D 3.3.1: $X \subseteq \mathbb{R}^n$ ***open*** if for any $x = (x_1,\dots,x_n) \in X \;\; \exists\delta > 0$ s.t. $\{y = (y_1,\dots,y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \;\; \forall i\}$ is contained in X. ($=$ changing a coordinate of x by $< \delta \to x' \in X$)

P 3.3.2: $X \subseteq \mathbb{R}^n$ open $\Leftrightarrow$ ***complement*** $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed

C 3.3.3: If $f : \mathbb{R}^n \to \mathbb{R}^m$ cont. and $Y \subseteq \mathbb{R}^m$ open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$

Ex 3.3.4: ***(1)*** $\emptyset$ and $\mathbb{R}^n$ are both open and closed. ***(2)*** Open ball $D = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$ is open in $\mathbb{R}^n$ (x_0 the center and r radius) ***(3)*** $I_1 \times \cdots \times I_n$ is open in $\mathbb{R}^n$ for I_i open ***(4)*** $X \subseteq \mathbb{R}^n$ open $\Leftrightarrow \forall x \in X \exists\delta > 0$ s.t. open ball of center x and radius δ is contained in X

D 3.3.5: *(Partial derivative)* Let $X \subseteq \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$ and $1 \leq i \leq n$. Then f has partial derivative on X with respect to the i-th variable (or coordinate), if $\forall x_0 = (x_{0,1},\dots,x_{0,n}) \in X$, $g(t) = f(x_{0,1},\dots,x_{0,i-1},t,x_{0,i+1},x_{0,n})$ on set $I = \{t \in \mathbb{R} : (x_{0,1},\dots,x_{0,i-1},t,x_{0,i+1},\dots,x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. The derivative $g'(x_{0,i})$ at $x_{0,i}$ is denoted: $\frac{\partial f}{\partial x_i}(x_0), \partial_{x_i} f(x_0)$ or $\partial_i f(x_0)$

P 3.3.7: Let $X \subseteq \mathbb{R}^n$ open, $f, g : X \to \mathbb{R}^m$ and $1 \leq i \leq n$. Then: ***(1)*** If f & g have ∂_i on X, then so does $f + g$ and $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$ ***(2)*** If $m = 1$ (i.e. $\mathbb{R}^1$) and f & g have ∂_i on X, then so does fg and $\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g)$ and if $g(x) \neq 0 \;\; \forall x \in X$, then if $f \div g$ has ∂_i on X, then so does $f \div g$ and $\partial_{x_i}(f \div g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g)) \div g^2$

D 3.3.8: *(Jacobi Matrix J)* Element $J_{ij} = \partial_{x_j} f_i(x)$ for function $f : X \to \mathbb{R}^m$ with $X \subseteq \mathbb{R}^n$ open. x_j is the j-th variable, f_i is the i-th component of the equation (i.e. in the vector of the function). J has m rows and n columns.

D 3.3.10: *(Gradient, Divergence)* for $f : X \to \mathbb{R}$ with $X \in \mathbb{R}^n$ open, the ***gradient*** is given by $\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$ and the trace of the Jacobi Matrix, $\mathrm{div}(f)(x_0) = \mathrm{Tr}(J_f(x_0)) = \sum_{i=1}^n \partial_{x_i} f_i(x_0)$ is called the ***divergence*** of f at x_0.
The gradient is simply the transpose of the Jacobian and it points in the direction of the ***steepest ascent***.

3.4 The differential

D 3.4.2: *(Differentiable function)* We have function $f : X \to \mathbb{R}^m$, linear map $u : \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in X$. f is differentiable at x_0 with differential u if $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} \dfrac{f(x) - f(x_0) - u(x - x_0)}{||x - x_0||} = 0$ where the limit is in $\mathbb{R}^m$. We denote $\mathrm{d}f(x_0) = u$. If f is differentiable at every $x_0 \in X$, then f is differentiable on X.

P 3.4.4: Let $f : X \to \mathbb{R}^m$ be differentiable on X

- f is continuous on X
- f admits partial derivatives on X with respect to each variable
- Assume $m = 1$, let $x_0 \in X$ and let $u(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n$ be diff. of f at x_0. Then $\partial_{x_i} f(x_0) = a_i$ for $1 \leq i \leq n$

P 3.4.6: Let $f, g : X \to \mathbb{R}^m$ with $X \subseteq \mathbb{R}^n$ open

- The function $f + g$ is differentiable with differential $\mathrm{d}(f + g) = \mathrm{d}f + \mathrm{d}g$. If $m = 1$, then fg is differentiable
- If $m = 1$ and if $g(x) \neq 0 \forall x \in X$, then $f \div g$ is differentiable

P 3.4.7: If f as above has all partial derivatives on X and if they are all continuous on X, then f is differentiable on X. The ***differential is the Jacobi Matrix of*** f ***at*** x_0. This implies that most elementary functions are differentiable.

P 3.4.8: *(Chain Rule)* For $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ both open and $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ are both differentiable. Then $g \circ f$ is differentiable on X and for any $x \in X$, its differential is given by $\mathrm{d}(g \circ f)(x_0) = \mathrm{d}g(f(x_0)) \circ \mathrm{d}f(x_0)$. The Jacobi matrix is $J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$ (RHS is a matrix product, i.e. multiply rows of first with cols of second matrix)

D 3.4.11: *(Tangent space)* The graph of the affine linear approximation $g(x) = f(x_0) + u(x - x_0)$, or the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x_0) + u(x - x_0)\}$$

D 3.4.13: *(Directional derivative)* f has a directional derivative $w \in \mathbb{R}^m$ in the direction of $v \in \mathbb{R}^n$, if the function g defined on the set $I = \{t \in \mathbb{R} : x_0 + tv \in X\}$ by $g(t) = f(x_0 + tv)$ has a derivative at $t = 0$ and is equal to w

R 3.4.14: Because X is open, the set I contains an open interval $]-\delta, \delta[$ for some $\delta > 0$.

P 3.4.15: Let f as previously be differentiable. Then for any $x \in X$ and non-zero $v \in \mathbb{R}^n$, f has a directional derivative at x_0 in the direction of v, given by $\mathrm{d}f(x_0)(v)$

R 3.4.16: The values of the above directional derivative are linear with respect to the vector v. Suppose we know the dir. der. w_1 and w_2 in directions v_1 and v_2, then the directional derivative in direction $v_1 + v_2$ is $w_1 + w_2$

Computing a directional derivative Always normalize the vector! We can compute a directional derivative using the differential $\lim\limits_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$ or using a 1-dimensional helper function $g : h \mapsto f(x_0 + hv)$, calculating the derivative of it and evaluating $g'(0)$. That corresponds to the directional derivative. E.g. for function $f : x, y \mapsto x^2 + y^2$, we have $g : h \mapsto (x_0 + h)^2 + (y_0 + h)^2$. A final option is to compute it using a matrix-vector product: $D_v f(x_0) = J_f(x_0) v$

Red arrows indicate no implication

$$\begin{tikzcd}
f \text{ cont. diff.} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{All } \partial_j f_i \text{ continuous} \arrow[d, Rightarrow] \\
f \text{ differentiable} \arrow[r, shift left, Rightarrow] \arrow[d, Rightarrow] & \text{All } \partial_j f_i \text{ exist} \arrow[l, shift left, Rightarrow, red] \arrow[dl, Rightarrow, red] \\
f \text{ continuous} &
\end{tikzcd}$$

3.5 Higher derivatives

D 3.5.1: f is in class C^1 if f is differentiable and all its partial derivatives are continuous. f is of class C^k if it is differentiable and each of its partial derivatives are in C^{k-1}. If $f \in C^k(X; \mathbb{R}^m)$ for all $k \geq 1$, then $f \in C^\infty(X; \mathbb{R}^m)$

P 3.5.4: *(Mixed derivatives commute)* $\partial_{x,y} f = \partial_{y,x}$, as well as $\partial_{x,y,z} = \partial_{x,z,y} = \ldots$, etc (all mixed derivatives commute) Since we have symmetry, we can use the notation $\partial_{x_1^{m_1}, \ldots, x_n^{m_n}} f = \frac{\partial^k}{\partial x^m} f = D^m f = \partial^m f$, where $m = (m_1, \ldots, m_n)$ and $m_1 + \ldots + m_n = k$. There are $\binom{n+k-1}{k}$ possible values for m and e.g. $(1, 1, 2)$ corresponds to the derivative $\frac{\partial^4 f}{\partial x \partial y \partial^2 z}$

R 3.5.6: Due to linearity of the partial derivative $\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$

Ex 3.5.8: *(Laplace operator)* $f \in C^2(X)$, $\nabla f \in C_1(X; \mathbb{R}^n)$, so $\mathrm{div}(\nabla f) = \sum_{i=1}^n \dfrac{\partial}{\partial_{x_i}} \left(\dfrac{\partial f}{\partial_{x_i}} \right) = \sum_{i=1}^n \dfrac{\partial^2 f}{\partial x_i^2}$ (called ***Laplacian***, Δf)

D 3.5.9: *(Hessian)* $f : X \to \mathbb{R}$ in C^2. For $x \in X$, the ***Hessian matrix*** of f at x is the symmetric square matrix

$$\mathrm{Hess}_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n} = H_f(x) \qquad (i\text{-th row, } j\text{-th column})$$

3.6 Change of variable

The idea is to substitute variables for others that make the equation easier to solve. A common example is to switch to polar coordinates from cartesian coordinates, as already demonstrated with continuity checks

3.7 Taylor polynomials

D 3.7.1: *(Taylor polynomials)* Let $f : X \to \mathbb{R}$ with $f \in C^k(X, \mathbb{R})$ and $y \in X$. The Taylor-Polynomial of order k of f at y is:

$$T_k f(y; x-y) = \sum_{|i| \leq k} \frac{\partial_i f(y)(x-y)^i}{i!}$$

where i is a *multi-index*, so:

- $i = (i_1, \ldots, i_n)$ (each $i_j \geq 0$)
- $\partial_i = \partial_1^{i_1} \ldots \partial_n^{i_n}$
- $i! = i_1! \cdot \ldots \cdot i_n!$
- $|i| = i_1 + \ldots + i_n$
- $(x-y)^i = (x_1 - y_1)^{i_1} \cdot \ldots \cdot (x_n - y_n)^{i_n}$

In the input, we have the vector y, which is the evaluation point, as well as the vector $x - y$ (where y is the evaluation point again and $x = (x_1, \ldots, x_n)$)

The concept this formula uses is that we iterate through all possible partial derivatives of f and assigns each a multi-index i. Do note that the formula expands to $f(y) + \ldots$, so also include the original function in the sum!

To denote that we want to take the partial derivative ∂_{112}, we use $i = (2, 1, 0)$, since we take the derivative of the first variable twice, of the second variable once and never of the third variable. This is also the explanation for what the $\partial_1^{i_1}$ means (we take the derivative regarding the first variable i_1 times, etc).

One of the elements of the sum (element with $i = (2, 1, 0)$) is for example:

$$\frac{\partial_{112} f(y)(x_1 - y_1)^2 (x_2 - y_2)^1 (x_3 - y_3)^0}{2!1!0!} = \frac{\partial_{112} f(y)(x_1 - y_1)^2 (x_2 - y_2)}{2}$$

3.8 Critical points

D 3.8.2: *(Critical Point)* For $f : X \to \mathbb{R}^n$ differentiable, $x_0 \in X$ is called a ***critical point*** of f if $\nabla f(x_0) = 0$

R 3.8.3: As in 1 dimensional case, check edges of the interval for the critical point.
To determine the kind of critical point, we need to determine if $H_f(x_0)$ is definite:

- positive definite $\Rightarrow x_0$ local max
- negative definite $\Rightarrow x_0$ local min
- indefinite $\Rightarrow x_0$ point of inflection

D 3.8.6: *(Non-degenerate critical point)* If $\det(H_f(x_0)) \neq 0$ (if $H_f(x_0)$ is semi-definite, then $\det(H_f(x_0)) = 0$, thus degenerate)

To figure out if a matrix is definite, we can compute the eigenvalues. A is positive (negative) definite, if and only if all eigenvalues are greater (lower) than 0. A is indefinite if and only if it has both positive and negative eigenvalues. A is positive (negative) semi-definite if and only if all eigenvalues are greater (lower) or equal to 0. It is positive (negative) definite if and only if all eigenvalues are greater (lower) than 0 (Compute Eigenvalues using $\det(A - \lambda I) = 0$)

For 2×2 matrices (i.e. 2D functions), we can use the following scheme (remember that the trace is the sum of the diagonal entries):

$$\begin{tikzcd}
\text{pos. semi-def.} & & & & \text{pos. def.} \\
& \mathrm{Tr}(A) \arrow[ul, "\text{positive}"'] \arrow[dl, "\text{negative}"] \arrow[d, "0"] & \det(A) \arrow[l, "0"'] \arrow[r, "\text{positive}"] \arrow[d, "\text{negative}"] & \mathrm{Tr}(A) \arrow[ur, "\text{positive}"] \arrow[dr, "\text{negative}"'] & \\
\text{neg. semi-def.} & A \text{ is zero} & \text{indefinite} & & \text{neg. def.}
\end{tikzcd}$$

As in Analysis I, it is important to also check the boundaries for maximums and minimums. For that, formulate formulas for the borders and check them for critical points.

4 Integral Calculus in Vector Space

4.1 Line integrals

D 4.1.1: Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. $f : I \to \mathbb{R}$ with $f(t) = (f_1(t), \ldots, f_n(t))$ continuous (also f_i cont.).

(1) Then $\int_a^b f(t)\ \mathrm{d}t = \left(\int_a^b f_1(t), \ldots, \int_a^b f_n(t)\right)$

(2) ***Parametrized Curve*** in $\mathbb{R}^n$ is a continuous map $\gamma : I \to \mathbb{R}^n$, piecewise in C^1, i.e. for $k \geq 1$, we have partition $a = t_0 < t_1 < \ldots < t_k = b$, such that if f is restricted to interval $]t_{j-1}, t_j[$, restriction is C^1. γ is a *path* between $\gamma(a)$ and $\gamma(b)$

(3) ***Line integral*** $X \subseteq \mathbb{R}^n$ is the image of γ, which is a parametrized curve and $f : X \to \mathbb{R}^n$ continuous

Integral $\int_a^b f(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t \in \mathbb{R}$ is line integral of f along γ, denoted $\int_\gamma f(s)\ \mathrm{d}s$ or $\int_\gamma f(s)\ \mathrm{d}\vec{s}$ or $\int_\gamma \omega$,

with $\omega = f_1(x)\ \mathrm{d}x_1 + \ldots f_n(x)\ \mathrm{d}x_n$

We usually call $f : X \to \mathbb{R}^n$ a ***vector field***, which maps each point $x \in X$ to a vector in $\mathbb{R}^n$, displayed as originating from x Often, we use V instead of f to denote the vector field. Ideally, to compute a line integral, we compute the derivative of γ and $V(\gamma(t))$ separately, then simply do the integral after. Be careful with hat functions like $|x|$, we need two separate integrals for each side of the center! Alternatively to using a line integral, see section 4.5 for a faster way

D 4.1.4: *(Oriented reparametrization)* of γ is parametrized curve $\sigma : [c, d] \to \mathbb{R}^n$ s.t $\sigma = \gamma \circ \varphi$, with $\varphi : [c, d] \to I$ cont. map, differentiable on $]a, b[$ and for which $\varphi(a) = c$ and $\varphi(b) = d$. Conversely, $\gamma = \sigma \circ \varphi^{-1}$

P 4.1.5: For $f : X \to \mathbb{R}^n$ with X containing the image of of γ and equivalently σ, we have $\int_\gamma f(s) \cdot\ \mathrm{d}\vec{s} = \int_\sigma f(s) \cdot\ \mathrm{d}\vec{s}$

D 4.1.8: *(Conservative Vector Field)* If for any $x_1, x_2 \in X$ the line integral $\int_\gamma f(s)\ \mathrm{d}\vec{s}$ is of the independent choice of γ in X

R 4.1.9: f conservative iff $\int_\gamma f(s)\ \mathrm{d}\vec{s} = 0$ for a *closed* ($\gamma(a) = \gamma(b)$) parametrized curve

T 4.1.10: Let X be open set, f conservative vector field. Then $\exists\ C^1$ function g s.t. $f = \nabla g$. If any two points of X can be joined by a parametrized curve, then g is unique up to a constant: if $\nabla g_1 = f$, then $g - g_1$ is constant on X

R 4.1.11: Two points $x, y \in X$ can be joined by parametrized curve γ if $\gamma(a) = x$ and $\gamma(b) = y$. In that case, X is called ***path-connected***. It is true when X is *convex* (e.g. when X is a disc or a product of intervals). If f is a vector field on X, then g is called a ***potential*** for f and it is not unique, since we can add a constant to g without changing the gradient.

P 4.1.13: For a vectorfield to be conservative, a *necessary condition* is that $\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{x_i}$ for any $1 \leq i \neq j \leq n \in \mathbb{N}$

D 4.1.15: *(Start Shaped Set)* $X \subseteq \mathbb{R}^n$ is star shaped if $\exists x_0 \in X$ s.t. $\forall x \in X$, the line segment from x to x_0 is contained in X, and we also say that X is *star shaped around* x_0

T 4.1.17: Let X start shaped and open, f a C^1 vector field fulfilling Proposition 4.1.13. Then f is conservative.

D 4.1.20: *(Curl)* Let $X \subseteq \mathbb{R}^3$ open and f a C^1 vector field. The ***curl*** of f is the conservative vector field $\mathrm{curl}(f) = \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix}$

Below a chart to figure out some properties:

$$\begin{array}{ccc} f = \nabla g \Leftrightarrow f \text{ conservative} \Leftrightarrow \int_\gamma f(s)\ \mathrm{d}\vec{s} = 0 \forall \text{ closed } \gamma \\ \scriptstyle\text{if } x \text{ start-shaped} \Downarrow \ \Uparrow \\ J_f \text{ symmetric} \underset{n=3}{\Leftrightarrow} \mathrm{curl}(f) = 0 \end{array}$$

4.2 Riemann integral in Vector Space

The integral of a continuous function $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^n$ bounded and closed, is denoted $\int_X f(x)\ \mathrm{d}x$ with properties:

(1) ***(Compatibility)*** If $n = 1$ and $X = [a, b]$, integral is the indefinite integral as per Analysis I

(2) ***(Linearity)*** If f, g are continuous on X and $a, b \in \mathbb{R}$, then $\int_X (af(x) + bg(x))\ \mathrm{d}x = a \int_X f(x)\ \mathrm{d}x + b \int_X g(x)\ \mathrm{d}x$

(3) ***(Positivity)*** If $f \leq g$, then so is the integral and if $f \geq 0$, so is the integral and if $Y \subseteq X$, then int. over Y is $\leq$ over X

(4) ***(Upper bound & Triangle Inequality)*** $\left|\int_X f(x)\ \mathrm{d}x\right| \leq \int_X |f(x)|\ \mathrm{d}x$ and $\left|\int_X (f(x) + g(x))\ \mathrm{d}x\right| \leq \int_X |f(x)|\ \mathrm{d}x \int_X |g(x)|$

(5) ***(Volume)*** The integral of f is the volume of $\{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\} \subseteq \mathbb{R}^{n+1}$. If X is a bounded rectangle, e.g. $X = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ and $f = 1$, then $\int_X\ \mathrm{d}x = (b_n - a_n) \ldots (b_1 - a_1)$. We write $\mathrm{Vol}(X)$ or $\mathrm{Vol}_n(X)$

(6) ***(Multiple integral)*** *(Fubini)* If $n_1, n_2 \in \mathbb{Z}$ s.t. $n = n_1 + n_2$, then for $x_1 \in \mathbb{R}^{n_1}$, let $Y_{x_1} = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in X\} \subseteq \mathbb{R}^{n_2}$. Let X_1 be the set of $x_1 \in \mathbb{R}^n$ such that Y_{x_1} is not empty. Then X_1 and Y_{x_1} are compact.

If $g(x_1) = \int_{Y_{x_1}} f(x_1, x_2)\ \mathrm{d}x_2$ is continuous on X_1, then

$$\int_X f(x_1, x_2)\ \mathrm{d}x = \int_{X_1} g(x_1)\ \mathrm{d}x = \int_{X_1} g(x_1)\ \mathrm{d}x_1 = \int_{X_1} \left(\int_{Y_{x_1}} f(x_1, x_2)\ \mathrm{d}x_2\right)\ \mathrm{d}x_1$$

Exchanging the role of x_1 and x_2 we have (with $Z_{x_2} = \{x_1 : (x_1, x_2) \in X\}$) if integral over x_1 is continuous.

$$\int_X f(x_1, x_2)\ \mathrm{d}x = \int_{X_2} \left(\int_{Z_{x_2}} f(x_1, x_2)\ \mathrm{d}x_1\right)\ \mathrm{d}x_2$$

(7) ***(Domain additivity)*** If X_1 and X_2 are compact and f continuous on $X = X_1 \cup X_2$, then (for $Y = X_1 \cap X_2$)

$$\int_X f(x)\ \mathrm{d}x + \int_Y f(x)\ \mathrm{d}x = \int_{X_1} f(x)\ \mathrm{d}x + \int_{X_2} f(x)\ \mathrm{d}x$$

In particular, if Y empty (or size is "negligible"), then $\int_X f(x)\ \mathrm{d}x = \int_{X_1} f(x)\ \mathrm{d}x + \int_{X_2} f(x)\ \mathrm{d}x$

D 4.2.3: For $m \leq n \in \mathbb{N}$, a ***parametrized*** m***-set*** in $\mathbb{R}^n$ is a continuous map $f : [a_1, b_1] \times \ldots \times [a_m, b_m] \to \mathbb{R}^n$, which is C^1 on $]a_1, b_1[\times \ldots \times]a_m, b_m[$. $B \subseteq \mathbb{R}^n$ is ***negligible*** if $\exists k \geq 0 \in \mathbb{Z}$ and parametrized m_i-sets $f_i : X_i \to \mathbb{R}^n$ with $1 \leq i \leq k$ and $m_i < n$ s.t. $X \subseteq f_1(x_1) \cup \ldots \cup f_k(X_k)$. A parametrized 1-set in $\mathbb{R}^n$ is a parametrized curve. **Ex 4.2.4:** Any $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ is negligible in $\mathbb{R}^2$, or more generally, if $H \subseteq \mathbb{R}^n$ is an affine subspsace of dimension $m < n$, then any subset of $\mathbb{R}^n$ that is contained in H is negligible. Image of par. curve $\gamma : [a, b] \to \mathbb{R}^n$ is negligible, since γ is a 1-set in $\mathbb{R}^n$

P 4.2.5: X compact set, negligible. Then for any cont. function on X, $\displaystyle\int_X f(x)\ \mathrm{d}x = 0$

Computing it ***How to find the actual integrals from the intervals:*** (Be careful with order of x and y!)

- Given an integral $\int_D f(x, y)\ \mathrm{d}x\ \mathrm{d}y$ for a set (or region) X that is bounded by the coordinate axes and the line $x + y = 2$, the integral we can actually compute is $\int_0^2 \int_0^2 f(x, y)\ \mathrm{d}y\ \mathrm{d}x$.
- Given an integral $\int_X g(x, y)\ \mathrm{d}x\ \mathrm{d}y$ with $X = [0, 1] \times [0, 2]$ and $g(x) = x^2 + y^2$. Then the integral should be obvious: $\int_0^1 \int_0^2 g(x, y)\ \mathrm{d}y\ \mathrm{d}x$
- *Harder example* Given integral $\int_Y h(x, y)\ \mathrm{d}x\ \mathrm{d}y$ with $Y = \{(x, y) \mid x \in [0, 1], y \leq 2x \wedge y \geq -2x\}$. A good idea is to visualize the set: This one is a triangle and the integral is $\int_0^1 \int_{-2x}^{2x} h(x, y)\ \mathrm{d}y\ \mathrm{d}x$
- *Non-obvious example* For a set $U = \{(x, y) : \sqrt{x^2 + y^2} \leq R\}$, we have the integral $\displaystyle\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ \mathrm{d}y\ \mathrm{d}x$. The new limits were attained by a simple inequality transformation, because in such equations, y could be 0 (and thus $|x|$ is limited by R)

How to compute the integral: We compute each integral "inside out". For a definite integral, don't just find the anti-derivative, compute the actual integral! For an integral as seen in the harder example, we compute it as we normally would, simply using the $\pm 2x$ as the a and b.

Using a change of variables into polar coordinates may come in handy, e.g. for a set like $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$, we can use polar coordinates and the integral is then $\int_0^{2\pi} \int_1^2 f(x, y)\ \mathrm{d}r\ \mathrm{d}\varphi$ (or flipped of course)

4.3 Improper integrals

As in the one-dimensional case, we are looking at integrals that are undefined at the edge of the interval and thus, we apply a limit to them, thus approaching said edge of the interval.

For example, in the two-dimensional case, disc $D_R = [-R, R]^2$ with radius R

$$\lim_{R \to \infty} \int_{D_R} f(d, y)\ \mathrm{d}x\ \mathrm{d}y$$

4.4 Change of Variable Formula

T 4.4.1: *(Change of variable formula)* $\overline{X}, \overline{Y} \subseteq \mathbb{R}^n$ compact, $\phi : \overline{X} \to \overline{Y}$ continuous. For the open sets X, Y, negligible sets B, C and restriction of $\phi : X \to Y$ to open set X is a C^1 bijection, we can write $\overline{X} = X \cup B$ and $\overline{Y} = Y \cup C$. The Jacobian $J_\phi(x)$ is invertible at all $x \in X$. For any continuous function f on $\overline{Y}$ we have $\displaystyle\int_{\overline{Y}} f(y)\ \mathrm{d}y = \int_{\overline{X}} f(\phi(x)) |\det(J_\phi(x))|\ \mathrm{d}x$

Computing the determinant Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$. For 3D: $B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and $\det(B) = a_1 \cdot b_2 \cdot c_3 + b_1 \cdot c_2 \cdot a_3 + c_1 \cdot a_2 \cdot b_3 - c_1 \cdot b_2 \cdot a_3 - b_1 \cdot a_2 \cdot c_3 - a_1 \cdot c_2 \cdot b_3$

How to use it We could use it for example to switch from Cartesian to polar coordinates (then $x = r \cdot \cos(\varphi)$ and $y = r \cdot \sin(\varphi)$).

Finding ϕ***:*** Given integral $\displaystyle\int_B (1 - x^2 - y^2)^{\frac{n-2}{2}}\ \mathrm{d}x\ \mathrm{d}y$ with $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Here, it should immediately ring a bell that this can be rewritten using polar coordinates with $x^2 + y^2$ simplifying to r^2. Thus, $\phi(r, \phi) = (r \cos(\varphi), r \sin(\varphi))$. The boundaries then have to be determined from the reference boundaries using the inverse function of ϕ

Computing the integral: When applying the formula, we replace all variables with their counterparts in ϕ (see above how to), we change the integration boundaries to fit our new variables and finally multiply everything by the Jacobian of ϕ

Example: Using the integral from above, we get:

$$\int_0^1 \int_0^{2\pi} (1 - (r\cos(\varphi))^2 - (r\sin(\varphi))^2)^{\frac{n-2}{2}} \cdot r\ \mathrm{d}\varphi\ \mathrm{d}r = \int_0^1 \int_0^{2\pi} (1 - r^2)^{\frac{n-2}{2}} \cdot r\ \mathrm{d}\varphi\ \mathrm{d}r$$

Likeliest case: Changing into polar coordinates, then we replace $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ and replace $\mathrm{d}x\ \mathrm{d}y = r\ \mathrm{d}r\ \mathrm{d}\varphi$

Connection to Analysis I: This is just the generalization of the substitution rule for integrals

4.5 The Green Formula

D 4.5.1: *(Simple parametrized curve)* $\gamma : [a, b] \to \mathbb{R}^2$ is a closed parametrized curve s.t. $\gamma(t) \neq \gamma(s)$ (if $s \neq t$ and $\{s, t\} = \{a, b\}$), s.t. $\gamma'(t) \neq 0$ for $a < t < b$. If γ only piecewise in C^1 in $]a, b[$, then only apply when $\gamma'(t)$ exists.

T 4.5.3: *(Green's Formula)* $X \subseteq \mathbb{R}^2$ compact set with boundary $\partial X = \gamma_1 \cup \ldots \cup y_k$ with $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) : [a_i, b_i] \to \mathbb{R}^2$ a simple closed parametrized curve, with property that X lies "to the left" of tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. $f = (f_1, f_2)$ is a vector field of class C^1 on open set containing X. Then:

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial_y}\right) \,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot \,\mathrm{d}\vec{s}$$

Corollary 4.5.5: $X \subseteq \mathbb{R}^2$ compact with boundary ∂X as before. γ_i as above, then

$$\mathrm{Vol}(X) = \sum_{i=1}^{k} \int_{\gamma_i} x \,\mathrm{d}\vec{s} = \sum_{i=1}^{k} \int_{a_i}^{b_i} \gamma_{i,1}(t)\gamma_{i,2}'(t) \,\mathrm{d}t$$

Understanding and applying Green's Formula The $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{y} = \mathrm{curl}(f)$, i.e. it is the 2D-curl of f. Thus, the sum of all line integrals is the same thing as the Riemann-Integral of the curl.

We can use Green's Formula to compute integrals. For that we need the set of curves that define the set. For the unit circle, that is just one curve, being $\gamma(t) = \begin{pmatrix} R \cdot \cos(t) \\ R \cdot \sin(t) \end{pmatrix}$, with $t \in [0, 2\pi]$. We then use the curve as the vector $\vec{s}$ in Green's Formula. As a reminder, the vectors are multiplied with the dot product. If we just have one curve, there is no sum (i.e. the sum sums up all the integral of all curves)

Example: To compute the line integral of the vector field $f(x, y) = \begin{pmatrix} x + y \\ 3x + y^2 \end{pmatrix}$ over a complicated curve. Instead of computing the line integral, we can use Green's Formula to compute the curl over the set enclosed by the curve. This has the benefit that depending on the vector field, we won't even have to evaluate the integral:

$$\int_S \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \,\mathrm{d}x\,\mathrm{d}y = \int_S (3 - 1) \,\mathrm{d}x\,\mathrm{d}y = \int_S 2 \,\mathrm{d}x\,\mathrm{d}y = 2\left((2 \cdot 1) + \frac{1}{2}\pi\right) = 4 + \pi$$

for the set $S = \{(x, y) \mid x \in [0, 2], y \in [-1, 0]\} \cup \{(x, y) \mid (x - 1)^2 + y^2 \leq 1, y \geq 0\}$.

That set is derived from the image that is given for the line. Be cognizant of what direction the integral goes, if the set is on the right hand side of the curve, the final result has to be negated to change the direction of the integral. If the curve doesn't fully enclose the set, then we can simply compute the line integrals of the missing sections and subtract them from the final result.

Center of mass The center of mass of an object $\mathcal{U}$ is given by $\overline{x}_i = \frac{1}{\mathrm{Vol}(\mathcal{U})} \int_{\mathcal{U}} x_i \,\mathrm{d}x$.