# Analysis Cheat-Sheet

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## Contents

# 1 Fields

## 1.1 Real numbers

**T 1.1:** *(Lindemann)* There is no equation of form $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ with $a_i \in Q$ such that $x = \pi$ is a solution

**C 1.8:** *(Archimedic Principle)* Let $x \in \mathbb{R}$ with $x > 0$ and $y \in \mathbb{R}$. Then exists $n \in \mathbb{N}$ with $y \leq n \cdot x$

**Max, min, absolute value** — **Definition 1.10**

Let $x, y \in \mathbb{R}$. Then:

*(i)* $\max\{x, y\} = \begin{cases} x & \text{if } y \leq x \\ y & \text{if } x \leq y \end{cases}$ *(ii)* $\min\{x, y\} = \begin{cases} y & \text{if } y \leq x \\ x & \text{if } x \leq y \end{cases}$ *(iii)* The absolute value of $x \in \mathbb{R} : |x| = \max x, -x$

**Absolute value properties** — **Theorem 1.11**

*(i)* $|x| \geq 0 \quad \forall x \in \mathbb{R}$ *(ii)* $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$ *(iii)* $|x + y| \leq |x| + |y|$ *(iv)* $|x + y| \geq ||x| - |y||$

**T 1.12:** *(Young's Inequality)* $\forall \varepsilon > 0, \quad \forall x, y \in \mathbb{R}$ we have: $2|xy| \leq \varepsilon x^2 + \frac{1}{\varepsilon}y^2$

**Bounds** — **Definition 1.13**

*(i)* $c \in \mathbb{R}$ upper bound of $A$ if $\forall a \in A : a \leq c$. $A$ bounded from above if upper bound for $A$ exists
*(ii)* $c \in \mathbb{R}$ lower bound of $A$ if $\forall a \in A : a \leq c$. $A$ bounded from below if lower bound for $A$ exists
*(iii)* Element $m \in \mathbb{R}$ **maximum** of $A$ if $m \in A$ and $m$ upper bound of $A$
*(iv)* Element $m \in \mathbb{R}$ **minimum** of $A$ if $m \in A$ and $m$ lower bound of $A$

**Supremum & Infimum** — **Theorem 1.16**

*(i)* The least upper bound of a set $A$ bounded from above is called the ***Supremum*** and given by $c := \sup(A)$. It only exists if the set is upper bounded.
*(ii)* The greatest lower bound of a set $A$ bounded from below is called the ***Infimum*** and given by $c := \inf(A)$. It only exists if the set is lower bounded.

**Supremum & Infimum** — **Corollary 1.17**

Let $A \subset B \subset \mathbb{R}$

*(1)* If $B$ is bounded from above, we have $\sup(A) \leq \sup(B)$ *(2)* If $B$ is bounded from below, we have $\inf(B) \leq \inf(A)$

## 1.3 Complex numbers

**Operations**: $i^2 = -1$ (NOT $i = \sqrt{-1}$ bc. otherwise $1 = -1$). Complex number $z_j = a_j + b_j i$. *Addition, Subtraction* $(a_1 \pm a_2) + (b_1 \pm b_2)i$. *Multiplication* $(a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$. *Division* $\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} + \frac{a_2b_1 - a_1b_2}{b_1^2b_2^2}i$;

**Parts**: $\Re(a+bi) := a$ (Real part), $\Im(a+bi) := b$ (imaginary part), $|z| := \sqrt{a^2 + b^2}$ (modulus), $\overline{a + bi} := a - bi$ (complex conjugate);

**Polar coordinates**: $a+bi$ (normal form), $r \cdot e^{i\phi}$ (polar form). Transformation polar $\to$ normal: $r \cdot \cos(\phi) + r \cdot \sin(\phi)i$. Transformation normal $\to$ polar: $|z| \cdot e^{i \cdot \arcsin(\frac{b}{|z|})}$;

**Square root of negative number**: $\sqrt{-c} = ci$

**Fundamental Theorem of Algebra** — **Theorem 1.18**

Let $n \geq 1, n \in \mathbb{N}$ and let

$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0, \qquad a_j \in \mathbb{C}$$

Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that

$$P(z) = (z - z_1)(z - z_2) \ldots (z - z_n)$$

The set $\{z_1, \ldots, z_n\}$ and the multiplicity of the zeros $z_j$ are hereby uniquely determined

**Surjectivity** Given a function $f : X \to Y$, it is surjective, iff $\forall y \in Y, \exists x \in X : f(x) = y$ (continuous function)

**Injectivity** $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

## 2 Sequences And Series

### 2.1 Limits

**D 2.5:** A sequence $(a_n)_{n\geq 1}$ is ***converging*** if $\exists l \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ the set $\{n \in \mathbb{N}^* : a_n \notin ]l-\varepsilon, l+\varepsilon[\}$ is finite. Every convergent sequence is bounded. **L 2.7:** $(a_n)_{n\geq 1}$ converges to $l = \lim_{n\to\infty} a_n \Leftrightarrow \forall \varepsilon > 0 \;\; \exists N \geq 1$ such that $|a_n - l| < \varepsilon \;\; \forall n \geq N$

**T 2.9:** $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ converging, $a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$. Then:

- **(1)** $(a_n + b_n)_{n\geq 1}$ converging and $\lim_{n\to\infty}(a_n + b_n) = a + b$;
- **(2)** $(a_n \cdot b_n)$ converging and $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$;
- **(3)** If additionally $b_n \neq 0 \;\; \forall n \geq 1$ and $b \neq 0$, then $(a_n \div b_n)_{n\geq 1}$ converging and $\lim_{n\to\infty}(a_n \div b_n) = a \div b$;
- **(4)** If $\exists K \geq 1$ with $a_n \leq b_n \;\; \forall n \geq K \Rightarrow a \leq b$

### 2.2 Weierstrass Theorem

**D 2.1:** $(a_n)_{n\geq 1}$ ***monotonically increasing (decreasing)*** if $a_n \leq a_{n+1}$ $(a_n \geq a_{n+1})$ $\forall n \geq 1$

**T 2.2:** *(Weierstrass)* $(a_n)_{n\geq 1}$ monotonically increasing (decreasing) and bounded from above (below) converges to $\lim_{n\to\infty} a_n = \sup\{a_n : n \geq 1\}$ ($\lim_{n\to\infty} a_n = \inf\{a_n : n \geq 1\}$), called supremum and infimum respectively **Ex 2.7:** $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$

**L 2.8:** *(Bernoulli Inequality)* $(1+x)^n \geq 1 + n \cdot x \;\; \forall n \in \mathbb{N}, x > -1$

### 2.3 Limit Superior and limit inferior

We define for $(a_n)_{n\geq 1}$ two monotone sequences $b_n = \inf\{a_k : k \geq n\}$ and $c_n = \sup\{a_k : k \geq n\}$, then $b_n \leq b_{n+1} \;\; \forall n \geq 1$ and $c_{n+1} \leq c_n \;\; \forall n \geq 1$, our series are bounded and converge and we have $\liminf_{n\to\infty} a_n := \lim_{n\to\infty} b_n$ and $\limsup_{n\to\infty} a_n := \lim_{n\to\infty} c_n$. We also have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

### 2.4 Cauchy-Criteria (Convergence Tests)

**L 2.1:** $(a_n)_{n\geq 1}$ converges if and only if it is bounded and $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$
**T 2.2:** *(Cauchy-Criteria)* $(a_n)_{n\geq 1}$ converging $\Leftrightarrow \;\; \forall \varepsilon > 0 \;\; \exists N \geq 1$ such that $|a_n - a_m| \leq \varepsilon \;\; \forall n, m \geq N$

### 2.5 Bolzano-Weierstrass Theorem

**D 2.1:** *(Closed interval)* Subset $I \subseteq \mathbb{R}$ of form as seen below, with length $\mathcal{L}(I) = b - a$ (for *(1)*) or $\mathcal{L}(I) = +\infty$:
*(1)* $[a,b]$; $a \leq b$; $a, b \in \mathbb{R}$ *(2)* $[a, +\infty[$; $a \in \mathbb{R}$ *(3)* $]-\infty, a]$; $a \in \mathbb{R}$ *(4)* $]-\infty, +\infty[= \mathbb{R}$
An interval $I$ is closed $\Leftrightarrow$ for every converging sequence of elements of $I$ the limit is also in $I$

**T 2.6:** *(Cauchy-Cantor)* Let $I_1 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$ a sequence of closed intervals with $\mathcal{L}(I_i) < +\infty$. Then $\bigcap_{n\geq 1}^{\infty} I_n \neq \emptyset$. If additionally $\lim_{n\to\infty} \mathcal{L}(I_n) = 0$, then the set contains exactly one point. **T 2.7:** $\mathbb{R}$ is not countable

**D 2.8:** *(Subsequence of $(a_n)_{n\geq 1}$)* $(b_n)_{n\geq 1}$ where $b_n = a_{l(n)}$ and $l(n) \leq l(n+1) \;\; \forall n \geq 1$

**T 2.9:** *(Bolzano-Weierstrass)* Every bounded sequence has a convergent subsequence. Also: $\liminf_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n$

### 2.6 Sequences in other spaces than just real numbers

**D 2.1:** Sequences in $\mathbb{R}^d$ and $\mathbb{C}$ are noted the same as in $\mathbb{R}$
**D 2.2:** $(a_n)_{n\geq 1}$ in $\mathbb{R}^d$ is *converging* if $\exists a \in \mathbb{R}^d$ such that $\forall \varepsilon > 0 \;\; \exists N \geq 1$ with $||a_n - a|| \leq \varepsilon \;\; \forall n \geq N$
**T 2.3:** Let $b = (b_1, \ldots, b_n)$ (coordinates of $b$, since $b$ is a vector). Then $\lim_{n\to\infty} a_n = b \Leftrightarrow \lim_{n\to\infty} a_{n,j} = b_j \;\; \forall 1 \leq j \leq d$
**T 2.7:** $(a_n)_{n\geq 1}$ converges $\Leftrightarrow$ $(a_n)_{n\geq 1}$ is a Cauchy-Sequence; Every bounded sequence has a converging subsequence.

### 2.7 Series

**D 2.1:** *(Convergence of a series)* $\sum_{k=1}^{\infty} a_k$ converges if $(S_n)_{n\geq 1}$ (sequence of partial sums) converges, i.e. $\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} S_n$
**Ex 2.2:** *(Geometric Series)* Converges with limit $\frac{1}{1-q}$, and $s_n = a_1 \cdot \frac{1-q^n}{1-q}$ **Ex 2.3:** *(Harmonic Series)* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

**T 2.4:** Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be converging, $\alpha \in \mathbb{C}$. Then:

1. $\sum_{k=1}^{\infty}(a_k + b_k)$ converging and $\sum_{k=1}^{\infty}(a_k + b_k) = \left(\sum_{k=1}^{\infty} a_k\right) + \left(\sum_{k=1}^{\infty} b_k\right)$
2. $\sum_{k=1}^{\infty}(\alpha \cdot a_k)$ converging and $\sum_{k=1}^{\infty}(\alpha \cdot a_k) = \alpha \cdot \left(\sum_{k=1}^{\infty} a_k\right)$

**T 2.5:** *(Cauchy-Criteria)* A series $\sum_{k=1}^{\infty} a_k$ is converging $\Leftrightarrow \forall \varepsilon > 0\ \exists N \geq 1$ with $|\sum_{k=n}^{m} a_k| \leq \varepsilon\ \ \forall m \geq n \geq N$

**T 2.6:** $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0\ \ \forall k \in \mathbb{N}^*$ converges $\Leftrightarrow (S_n)_{n\geq 1}, S_n = \sum_{k=1}^{n} a_k$ is bounded from above

**C 2.7:** *(Comparison theorem)* $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ with $0 \leq a_k \leq b_k\ \ \forall k \geq K$ (where $K \geq 1$), then:

$$\sum_{k=1}^{\infty} b_k \text{ converging} \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ converging} \qquad \sum_{k=1}^{\infty} a_k \text{ diverging} \Longrightarrow \sum_{k=1}^{\infty} b_k \text{ diverging}$$

**D 2.9:** *(Absolute convergence)* A series for which $\sum_{k=1}^{\infty} |a_k|$ converges. Using the Cauchy-Criteria we get:

**T 2.10:** A series converging absolutely is also convergent and $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$

## Convergence tests

$\sum_{a=0}^{\infty} \frac{1}{a^p}$ converges for $n > 1$

**T 2.12:** *(Leibniz)* Let $(a_n)_{n\geq 1}$ monotonically decreasing with $a_n \geq 0\ \forall n \geq 1$ and $\lim\limits_{n\to\infty} a_n = 0$. Then $S := \sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges and $a_1 - a_2 \leq S \leq a_1$

**Usage** To show convergence, prove that $(a_n)_{n\geq 1}$ is monotonically decreasing, $a_n \geq 0$ and that the limit is 0

**D 2.15:** *(Reordering)* A series $\sum_{k=1}^{\infty} a'_k$ for a $\sum_{k=1}^{\infty} a_k$ if there is a bijection $\phi$ such that $a'_n = a_{\phi(n)}$

**T 2.17:** *(Dirichlet)* If $\sum_{k=1}^{\infty} a_k$ has absolute convergence, every reordering of the series converges to the same limit.

**T 2.18:** *(Ratio test)* Series $s$ with $a_n \neq 0\ \forall n \geq 1$, $s$ has absolute convergence if $\limsup\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$. If $\liminf\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ it diverges. If any of the two limits are 1, the test was inconclusive

**T 2.19:** *(Root test)* If $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ the series converges. If the limit is larger than one, it diverges

**C 2.20:** *(Radius of convergence)* A power series of form $\sum_{k=0}^{\infty} c_k z^k$ has absolute convergence for all $|z| < \rho$ and diverges for all $|z| > \rho$. Let $l = \limsup_{n\to\infty} \sqrt[k]{|c_k|}$, then $\rho = \begin{cases} +\infty & \text{if } l = 0 \\ \frac{1}{l} & \text{if } l > 0 \end{cases}$. The *radius of convergence* is then given by $\rho$ if $\rho \neq \infty$

## Double series

**D 2.23:** For a double series $\sum_{i,j\geq 0}^{\infty} a_{ij}$, $\sum_{k=0}^{\infty} b_k$ is a ***linear arrangement*** if there exists a bijection $\sigma$ s.t. $b_k = a_{\sigma(k)}$

**T 2.24:** *(Cauchy)* Assume $\exists B \geq 0$ s.t. $\sum_{i=0}^{m}\sum_{j=0}^{m} |a_{ij}| \leq B\ \ \forall m \geq 0$. Then: $S_i := \sum_{j=0}^{\infty} a_{ij}\ \ \forall i \geq 0$ and $U_j := \sum_{i=0}^{\infty} a_{ij}\ \ j \geq 0$ have absolute convergence, as well as $\sum_{i=0}^{\infty} S_i$ and $\sum_{j=0}^{\infty} U_j$ and we have: $\sum_{i=0}^{\infty} S_i = \sum_{j=0}^{\infty} U_j$.

Every linear double series has absolute convergence with same limit.

**D 2.25:** *(Cauchy-Product)* $\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j} b_j\right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \ldots$ for two series $\sum_{i=0}^{\infty} a_i$, $\sum_{j=0}^{\infty} b_j$

**T 2.27:** If two series have absolute convergence, their Cauchy-Product converges and it is the terms of the two series expanded.

**T 2.28:** Let $f_n$ be a sequence. We assume that:

- $f(j) := \lim\limits_{n\to\infty} f_n(j)$ exists $\forall j \in \mathbb{N}$
- $\exists g$ s.t. $|f_n(j)| \leq g(j)\ \ \forall j, n \geq 0$ and $\sum_{j=0}^{\infty} g(j)$ converges

Then $\sum_{j=0}^{\infty} f(j) = \lim\limits_{n\to\infty} \sum_{j=0}^{\infty} f_n(j)$

**C 2.29:** For every $z \in \mathbb{C}$ we have $\lim\limits_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = \exp(z)$ and it converges, where $\exp(z) := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$

# 3 Continuous Functions

## 3.1 Real-Valued functions

**D 3.1:** *(Bounds)* Let $f \in \mathbb{R}^D$, where $\mathbb{R}^D$ is the set of all functions $f : D \to \mathbb{R}$, which is a vector space

- $f$ is ***bounded from above*** if $f(D) \subseteq \mathbb{R}$ is bounded from above.
- $f$ is ***bounded from below*** if $f(D) \subseteq \mathbb{R}$ is bounded from below.
- $f$ is ***bounded*** if $f(D) \subseteq \mathbb{R}$ is bounded.

**D 3.2:** *(Monotonicity)* If $D \subseteq \mathbb{R}$ we have the following terms for monotonicity:

- ***monotonically increasing*** if $\forall x, y \in D \; x \leq y \Rightarrow f(x) \leq f(y)$
- ***strictly monotonically increasing*** if $\forall x, y \in D \; x < y \Rightarrow f(x) < f(y)$
- ***monotonically decreasing*** if $\forall x, y \in D \; x \leq y \Rightarrow f(x) \geq f(y)$
- ***strictly monotonically decreasing*** if $\forall x, y \in D \; x < y \Rightarrow f(x) > f(y)$
- ***monotone*** if $f$ is monotonically increasing or monotonically decreasing
- ***strictly monotone*** if $f$ is strictly monotonically increasing or strictly monotonically decreasing

## 3.2 Continuity

**Intuition:** we can draw a continuous function without lifting the pen.

**D 3.1:** *(Continuity of $f$ in $x_0$)* If for every $\varepsilon > 0$ exists a $\delta$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ **D 3.2:** *(Continuity)* $f$ continuous if continuous in all points of $D$ **T 3.4:** $f$ is continuous in $x_0 \iff$ for $(a_n)_{n\geq 1}$ $\lim_{n\to\infty} a_n = x_0 \Rightarrow f(a_n) = f(x_0)$

**C 3.5:** Let $f$, $g$ continuous in $x_0$, then $f + g$, $\lambda \cdot f$, $f \cdot g$, $f \circ g$ are continuous in $x_0$ and if $g(x_0) \neq 0$, $\frac{f}{g}$ is continuous in $x_0$ for $\frac{f}{g} : D \cap \{x \in D : g(x) \neq 0\} \to \mathbb{R}$

**D 3.6:** *(Polynomial function)* $P(x) = a_n x^n + \ldots + a_0$, if $a_n \neq 0$, $\deg(P) = n$ (degree of $P$) **C 3.7:** They are continuous on all of $\mathbb{R}$ **C 3.8:** $P, Q$ pol. func. on $\mathbb{R}$ with $Q \neq 0$, where $x_1, \ldots, x_m$ are zeros of $Q$. Then: $\frac{P}{Q} : \mathbb{R}\backslash\{x_1, \ldots, x_m\} \to \mathbb{R}$ is continuous

## 3.3 Intermediate value theorem

**T 3.1:** Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a continuous function and $a, b \in I$. For each $c$ between $f(a)$ and $f(b)$ exists a $z$ between $a$ and $b$ with $f(z) = c$ **C 3.2:** Let $P$ be a polynomial with $\deg(P) = n$, $n$ odd. Then, $P$ has *at least* one zero in $\mathbb{R}$

## 3.4 Min-Max-Theorem

**D 3.2:** *(Compact interval)* if interval $I$ is of form $I = [a, b]$, $a \leq b$ **L 3.3:** $f, g$ continuous in $x_0$. Then: $|f|$, $\max(f, g)$ and $\min(f, g)$ are continuous in $x_0$ ($\min(f, g)$ is the minimum of the two functions at each $x$) **L 3.4:** $(x_n)_{n\geq 1}$ converging series in $\mathbb{R}$ with $\lim_{n\to\infty} x_n \in \mathbb{R}$ and $a \leq b$. If $\{x_n : n \geq 1\} \subseteq [a, b]$ we have $\lim_{n\to\infty} x_n \in [a, b]$ **T 3.5:** Let $f$ continuous on compact interval $I$. Then $\exists u \in I$ and $\exists v \in I$ with $f(u) \leq f(x) \leq f(v) \;\; \forall x \in I$. $f$ is bounded.

## 3.5 Inverse function theorem

**T 3.1:** Let $D_1, D_2 \subseteq \mathbb{R}$, $f : D_1 \to D_2$, $g : D_2 \to \mathbb{R}$, $x_0 \in D_1$. If $f$ cont. in $x_0$, $g$ in $f(x_0)$ then $f \circ g : D_1 \to \mathbb{R}$ is continuous in $x_0$
**C 3.2:** If in theorem 3.5.1 $f$ continuous on $D_1$ and $g$ on $D_2$, then $g \circ f$ is continuous on $D_1$
**T 3.3:** *(Inverse function theorem)* Let $f : I \to \mathbb{R}$ continuous, strictly monotone and let $I \subseteq \mathbb{R}$ be an interval. Then: $J := f(I) \subseteq \mathbb{R}$ is an interval and $f^{-1} : J \to I$ continuous and strictly monotone.

## 3.6 Real-Valued exponential function

The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is usually given by a power series converging on all $\mathbb{C}$: $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$, here for $z \in \mathbb{R}$. exp is bijective, continuous, strictly monotonically increasing and smooth. $\exp^{-1}(x) = \ln(x)$
**T 3.1:** $\exp : \mathbb{R} \to ]0, +\infty[$ is strictly monotonically increasing, continuous and surjective **C 3.2:** $\exp(x) > 0 \;\; \forall x \in \mathbb{R}$
**C 3.3:** $\exp(z) > \exp(y) \;\; \forall z > y$ **C 3.4:** $\exp(x) \geq 1 + x \;\; \forall x \in \mathbb{R}$ **C 3.5:** $\ln : ]0, +\infty[ \to \mathbb{R}$ is strictly monotonically increasing, continuous and bijective. We have $\ln(a \cdot b) = \ln(a) + \ln(b) \;\; \forall a, b \in ]0, +\infty[$. It is the inverse function of exp **C 3.6:**

1. For $a > 0$ $\;]0, +\infty[ \; \to \; ]0, +\infty[ \;\; x \mapsto x^a$ is a continuous, strictly monotonically increasing bijection.
2. For $a < 0$ $\;]0, +\infty[ \; \to \; ]0, +\infty[ \;\; x \mapsto x^a$ is a continuous strictly monotonically decreasing bijection.
3. $\ln(x^a) = a \ln(x) \;\; \forall a \in \mathbb{R}, \;\; \forall x > 0$
4. $x^a \cdot x^b = x^{a+b} \;\; \forall a, b \in \mathbb{R}, \;\; \forall x > 0$
5. $(x^a)^b = x^{a \cdot b} \;\; \forall a, b \in \mathbb{R}, \;\; \forall x > 0$

## 3.7 Convergence of sequences of functions

**D 3.1:** *(Pointwise convergence)* $(f_n)_{n\geq 1}$ converges pointwise towards a function $f : D \to \mathbb{R}$ if for all $x \in D$ $f(x) = \lim_{n\to\infty} f_n(x)$

**D 3.3:** *(Weierstrass)* Sequence $f_n$ converges uniformly in $D$ to $f$ if $\forall \varepsilon > 0$ $\exists N \geq 1$ s.t. $\forall n \geq N$, $\forall x \in D : |f_n(x) - f(x)| < \varepsilon$

**T 3.4:** $f_n$ sequence of (in $D$) continuous functions converging to $f$ uniformly in $D$. Then, $f$ is continuous (in $D$)

**D 3.5:** *(Uniform convergence of $(f_n)_{n\geq 1}$))* $f_n$ if $\forall x \in D$ $f(x) := \lim_{n\to\infty} f_n(x)$ exists and $(f_n)_{n\geq 1}$ converges uniformly to $f$

**C 3.6:** $f_n$ converges uniformly in $D \iff \forall \varepsilon > 0$ $\exists N \geq 1$ such that $\forall n, m \geq N$, $\forall x \in D$ $|f_n(x) - f_m(x)| < \varepsilon$

**C 3.7:** If $f_n$ is a uniformly converging sequence of functions, then $f(x) := \lim_{n\to\infty} f_n(x)$ is continuous

**D 3.8:** $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly if $S_n(x) := \sum_{k=0}^{n} f_k(x)$ does **T 3.9:** Assume $|f_n(x)| \leq c_n$ $\forall x \in D$ and that $\sum_{n=0}^{\infty} c_n$ converges. Then $\sum_{n=0}^{f_n(x)}$ converges uniformly in $D$ and $f(x) := \sum_{n=0}^{\infty} f_n(x)$ is continuous in $D$

**D 3.10:** *(Radius of convergence)* See **C 2.7.19** **T 3.11:** A power series converges uniformly on $]-r, r[$ where $0 \leq r < \rho$

## 3.8 Trigonometric Functions

**T 3.1:** $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are continuous functions **T 3.2:**

1. $\exp iz = \cos(z) + i\sin(z)$ $\forall z \in \mathbb{C}$
2. $\cos(z) = \cos(-z)$ and $\sin(-z) = -\sin(z)$ $\forall z \in \mathbb{C}$
3. $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$; $\cos(z) = \dfrac{e^{iz} + e^{iz}}{2}$
4. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
   $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
5. $\cos(z)^2 + \sin(z)^2 = 1$ $z \in \mathbb{C}$

**C 3.3:** $\sin(2z) = 2\sin(z)\cos(z)$ and $\cos(2z) = \cos(z)^2 - \sin(z)^2$

## 3.9 Pie (delicious)

**T 3.1:** The sine function has at least one zero on $]0, +\infty[$ and $\pi := \inf\{t > 0 : \sin(t) = 0\}$. Then $\sin(\pi) = 0$, $\pi \in ]2, 4[$; $\forall x \in ]0, \pi[ : \sin(x) > 0$ and $e^{\frac{i\pi}{2}} = i$ **C 3.2:** $x \geq \sin(x) \geq x - \frac{x^3}{3!}$ $\forall 0 \leq 0 \leq \sqrt{6}$ **C 3.3:**

1. $e^{i\pi} = -1$, $e^{2i\pi} = 1$
2. $\sin\left(x + \frac{\pi}{2}\right)$, $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$ $\forall x \in \mathbb{R}$
3. $\sin(x + \pi) = -\sin(x)$, $\sin(x + 2\pi) = \sin(x)$ $\forall x \in \mathbb{R}$
4. $\cos(x + \pi) = -\cos(x)$, $\cos(x + 2\pi) = \cos(x)$ $\forall x \in \mathbb{R}$
5. Zeros of sine $= \{k \cdot \pi : k \in \mathbb{Z}\}$
   $\sin(x) > 0$ $\forall x \in ]2k\pi, (2k+1)\pi[$, $k \in \mathbb{Z}$ $\sin(x) > 0$ $\forall x \in ](2k+1)\pi, (2k+2)\pi[$, $k \in \mathbb{Z}$
6. Zeros of cosine $= \{\frac{\pi}{2} \cdot k \cdot \pi : k \in \mathbb{Z}\}$
   $\cos(x) > 0$ $\forall x \in ]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[$, $k \in \mathbb{Z}$
   $\cos(x) > 0$ $\forall x \in ]-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[$, $k \in \mathbb{Z}$

## 3.10 Limits of functions

**D 3.1:** *(Cluster point)*DE: "Häufungspunkt" $x_0 \in \mathbb{R}$ of $D$ if $\forall \delta > 0$ $(]x_0 - \delta, x_0 + \delta[\setminus\{x_0\}) \cap D \neq \emptyset$

**D 3.3:** $A \in \mathbb{R}$ is the limit of $f(x)$ for $x \to x_0$ denoted $\lim_{x\to x_0} f(x) = A$, where $x_0$ is a cluster point, if:

$$\forall \varepsilon \ \exists \delta > 0 \text{ s.t. } \forall x \in D \cap (]x_0 - \delta, x_0 + \delta[\setminus\{x_0\}) : |f(x) - A| < \varepsilon$$

**T 3.7:** Let $D, E \subseteq \mathbb{R}$, $x_r$ a cluster point of $D$ and $f : D \to E$ a function. Assume that $y_0 := \lim_{x\to x_0}$ exists and $y_0 \in E$. If $g : E \to \mathbb{R}$ is continuous in $y_0$, we have $\lim_{x\to x_0} g(f(x)) = g(y_0)$

### Left / Right hand limit

Used when we have functions with poles, we approach them from both sides to evaluate said pole. Differently from at Kanti, we note it $x \to x_0^-$ instead of $x \uparrow x_0$

# 4 Differentiable Functions

## 4.1 Differentiation

**D 4.1:** *(Differentiability)* $f$ is differentiable in $x_0$ if $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists.

**T 4.3:** $x_0$ cluster point of $D$: $f$ is differentiable in $x_0 \iff \exists c \in \mathbb{R}$ and $r : D \to \mathbb{R}$ with (if it applies $c = f'(x_0)$ is unique):

$$f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0) \text{ as well as } r(x_0) = 0 \text{ and } r \text{ is continuous in } x_0$$

**T 4.4:** $f$ differentiable in $x_0 \Leftrightarrow \exists \phi : D \to \mathbb{R}$ continuous in $x =$ and $f(x) = f(x_0) + \phi(x)(x - x_0)\ \forall x \in D$. Then $\phi(x_0) = f'(x_0)$

**C 4.5:** $x_0 \in D$ cluster point of $D$. If $f$ differentiable in $x_0$, $f$ continuous in $x_0$ **D 4.7:** $f$ is differentiable on all $D$ if for each cluster point $x_0$ it is differentiable in $x_0$

**T 4.10:** *(Basic Differentiation rules)* Let $f, g$ be functions differentiable in $x_0$

- $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- if $g(x_0) \neq 0$, $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$

**T 4.12:** *(Chain rule)* $x_0 \in D$ cluster point, $f : D \to E$ differentiable in $x_0$ s.t. $y_0 := f(x_0) \in E$ cluster point of $E$ and let $g : E \to \mathbb{R}$ differentiable in $y_0$. Then $g \circ f : D \to \mathbb{R}$ differentiable in $x_0$ and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

**C 4.13:** Let $f : D \to E$ be a bijective function, differentiable in $x_0$ (cluster point) and $f'(x_0) \neq 0$ as well as $f^{-1}$ continuous in $y_0 = f(x_0)$. Then $y_0$ cluster point of $E$, $f^{-1}$ differentiable in $y_0$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

## 4.2 First derivative: Important Theorems

**D 4.1:** **(1)** $f$ has maximum at $x_0$ if $\exists \delta > 0$ s.t. $f(x) \leq f(x_0)\ \forall x \in ]x_0 - \delta, x_0 + \delta[\ \cap\ D$ **(2)** $f$ has minimum at $x_0$ if $\exists \delta > 0$ s.t. $f(x) \geq f(x_0)\ \forall x \in ]x_0 - \delta, x_0 + \delta[\ \cap\ D$ **(3)** $f$ has extrema in $x_0$ if it is either max or min

**T 4.2:** Assume $f$ differentiable in $x_0$. From the following we have that if $f'(x_0) = 0$, there is an extrema at $x_0$

1. If $f'(x_0) > 0\ \ \exists \delta > 0$ s.t. $f(x) > f(x_0)\ \ \forall x \in ]x_0, x_0 + \delta[$ and $f(x) < f(x_0)\ \ \forall x \in ]x_0 - \delta, x_0[$
2. If $f'(x_0) < 0\ \ \exists \delta > 0$ s.t. $f(x) < f(x_0)\ \ \forall x \in ]x_0, x_0 + \delta[$ and $f(x) > f(x_0)\ \ \forall x \in ]x_0 - \delta, x_0[$

**T 4.3:** Let $f : [a, b] \to \mathbb{R}$ continuous and differentiable in $]a, b[$. If $f(a) = f(b)$, $\exists \xi \in ]a, b[$ with $f'(\xi) = 0$

**T 4.4:** Let $f$ as above, then $\exists \xi \in ]a, b[$ s.t. $f(b) - f(a) = f'(\xi)(b - a)$ **C 4.5:** Let $f, g$ as above ($I = [a, b]$), then:

1. $f'(\xi) = 0\ \ \forall \xi \in ]a, b[\ \Rightarrow f$ constant
2. $f'(\xi) = g'(\xi)\ \ \forall \xi \in ]a, b[\ \Rightarrow \exists c \in \mathbb{R}$ with $f(x) = g(x) + c\ \ \forall x \in [a, b]$
3. $f'(\xi) \geq 0\ \ \forall \xi \in ]a, b[\ \Rightarrow f$ mon. increasing on $I$
4. $f'(\xi) > 0\ \ \forall \xi \in ]a, b[\ \Rightarrow f$ strictly mon. inc. on $I$
5. $f'(\xi) \leq 0\ \ \forall \xi \in ]a, b[\ \Rightarrow f$ mon. decreasing on $I$
6. $f'(\xi) < 0\ \ \forall \xi \in ]a, b[\ \Rightarrow f$ strictly mon. dec. on $I$
7. If $\exists M \geq 0$ s.t. $|f'(\xi)| \leq M\ \ \forall \xi \in ]a, b[$, then $\forall x_1, x_2 \in [a, b]\ \ |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

**T 4.10:** $f, g, \xi$ as defined previously. Then $g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$. If $g'(x) \neq 0\ \ x \in ]a, b[$, $g(a) \neq g(b)$ and $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$ **T 4.11:** *(L'Hospital's rule)* $f, g$ as before, with $g'(x) \neq 0\ \ \forall x \in ]a, b[$. If $\lim_{x \to b^-} f(x) = 0$, $\lim_{x \to b^-} g(x) = 0$ and $\lambda := \lim_{x \to b^-} \frac{f'(x)}{g'(x)}$ exists, we have $\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{x \to b^-} \frac{f'(x)}{g'(x)}$ **D 4.14:** $f$ ***convex*** on $I$ if $\forall x \leq y \in I$ and $\lambda \in [0, 1]$ $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$. ***Strictly convex*** if $<$ instead of $\leq$ in all occurences **T 4.17:** $f$ (as usual) (strictly) convex $\iff f'$ (strictly) monotonically increasing. **C 4.18:** If $f''$ exists, then $f$ (strictly) convex if $f'' \geq 0$ (or $f'' > 0$) on $]a, b[$

## 4.3 Higher derivatives

**Higher derivatives** — **Definition 4.1**

1. For $n \geq 2$, $f$ differentiable $n$ times in $D$ if $f^{(n-1)}$ is differentiable in $D$. $f^{(n)} := (f^{(n-1)})'$, $n$-th derivative of $f$
2. $f$ is $n$-times continuously differentiable in $D$ if $f^{(n)}$ exists and is continuous in $D$
3. $f$ is called smooth (de: glatt) in $D$ if $\forall n \geq 1\ f^{(n)}$ exists.

**T 4.3:** ***(1)*** $(f + g)^{(n)} = f^{(n)} + g^{(n)}$, ***(2)*** $(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$ (binomial expansion), for $f, g$ differentiable $n$ times

**T 4.5:** $f, g$ as above; If $g(x) \neq 0\ \ \forall x \in D$, then $\frac{f}{g}$ differentiable $n$-times in $D$ **T 4.6:** Let $E, D \subseteq \mathbb{R}$ for which each point is a cluster point and $f : D \to E$ and $g : E \to D$, both differentiable $n$ times. Then $(g \circ f)^{(n)}(x) = \sum_{k=1}^{n} A_{n,k}(x)(g^{(k)} \circ f)(x)$ where $A_{n,k}$ is a polynomial in the functions $f', f^{(2)}, \ldots, f^{(n+1-k)}$

## 4.4 Power series and Taylor approximation

**T 4.1:** Assume that $(f_n)_{n\geq 1}$ (for $f_n$ and $f_n'$ continuously differentiable) and $(f_n')_{n\geq 1}$ converge uniformly on $]a,b[$ for $f:]a,b[\to \mathbb{R}$ with $f := \lim_{n\to\infty} f_n$ and $p := \lim_{n\to\infty} f_n'$. Then $f$ is continuously differentiable and $f' = p$

**T 4.2:** Power series $\sum_{k=0}^{\infty} c_k x^k$ with $\rho > 0$, $f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$ differentiable on $]x_0-\rho, x_0+\rho[$ and $f'(x) = \sum_{k=1}^{\infty} k c_k (x-x_0)^{k-1}$

**C 4.3:** As in 4.4.1, $f$ smooth on conv. interval and $f^{(j)}(x) \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(x-x_0)^{k-j}$. Specifically, $c_j = \frac{f^{(j)}(x_0)}{j!}$

**T 4.5:** $f$ continuous, $\exists f^{(n+1)}$. For each $a < x \leq b$ $\exists \xi \in ]a,x[$ with $f(x) \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ **C 4.6:** *(Taylor Approximation)* Same as above, but $f:[c,d]\to\mathbb{R}$ instead of $f:[a,b]\to\mathbb{R}$ and $c<a<d$ and $\xi$ between $x$ and $a$.

**C 4.7:** $a < x_0 < b$ and $f$ as before, assume that $f'(x_0) = f^{(2)}(x_0) = \ldots = f^{(n)}(x_0) = 0$. Then:

1. If $n$ even and $x_0$ local extrema, $f^{(n+1)}(x_0) = 0$
2. If $n$ odd and $f^{(n+1)}(x_0) > 0$, $x_0$ strict local minimum
3. If $n$ odd and $f^{(n+1)}(x_0) < 0$, $x_0$ strict local maximum

**C 4.8:** $f$ differentiable twice and $a < x_0 < b$, assume $f'(x_0) = 0$

1. $f^{(2)}(x_0) > 0$, $x_0$ strict local minimum
2. $f^{(2)}(x_0) < 0$, $x_0$ strict local maximum

## 4.5 Exercise Help

| | |
|---|---|
| $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ | $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ | $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ |
| $\sum_{i=1}^{\infty} \frac{1}{n(n+1)} = 1$ | $\sum_{i=1}^{\infty} z^i = \frac{1-z^{i+1}}{1-z}$ |

**Common limits**

| | |
|---|---|
| $\lim_{x\to\infty} \frac{1}{x} = 0$ | $\lim_{x\to\infty} 1 + \frac{1}{x} = 1$ |
| $\lim_{x\to\infty} e^x = \infty$ | $\lim_{x\to-\infty} e^x = 0$ |
| $\lim_{x\to\infty} e^{-x} = 0$ | $\lim_{x\to-\infty} e^{-x} = \infty$ |
| $\lim_{x\to\infty} \frac{e^x}{x^m} = \infty$ | $\lim_{x\to-\infty} xe^x = 0$ |
| $\lim_{x\to\infty} \ln(x) = \infty$ | $\lim_{x\to 0} \ln(x) = -\infty$ |
| $\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = 1$ | $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$ |
| $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^b = 1$ | $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^b = 1$ |
| $\lim_{x\to\infty} x^a q^x = 0,\ \forall 0 \leq q < 1$ | $\lim_{x\to\infty} n^{\frac{1}{n}} = 1$ |
| $\lim_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = \mathrm{e}$ | $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = \frac{1}{\mathrm{e}}$ |
| $\lim_{x\to\pm\infty} \left(1+\frac{k}{x}\right)^{mx} = e^{km}$ | $\lim_{x\to 0} \frac{\sin x}{x} = 1$ |
| $\lim_{x\to 0} \frac{1}{\cos(x)} = 1$ | $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ |
| $\lim_{x\to 0} \frac{\log 1-x}{x} = -1$ | $\lim_{x\to 0} x \log x = 0$ |
| $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ | $\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1$ |
| $\lim_{x\to 0} \frac{x}{\arctan x} = 1$ | $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$ |
| $\lim_{x\to\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}$ | $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ |
| $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)\ \forall a > 0$ | $\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a$ |
| $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$ | $\lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$ |
| $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ | $\lim_{x\to\infty} \frac{\log(x)}{x^a} = 0$ |
| $\lim_{x\to\infty} \sqrt[x]{x} = 1$ | $\lim_{x\to\infty} \frac{2x}{2^x} = 0$ |
| $\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$ | $\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$ |
| $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ | $\lim_{x\to 0+} x \ln x = 0$ |

**Common Taylor Polynomials**

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \mathcal{O}(x^8)$$

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \mathcal{O}(x^8)$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \mathcal{O}(x^4)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)$$

# 5 Integrals

## 5.1 Definition and integrability

**D 5.1:** *(Partition)* finite subset $P \subset I$ where $I = [a,b]$ and $\{a,b\} \subseteq P$

Lower sum: $s(f,P) := \sum_{i=1}^{n} f_i \delta_i, \quad f_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$, Upper sum: $S(f,P) := \sum_{i=1}^{n} f_i \delta_i, \quad f_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$, $\delta_i$ sub-interval

**L 5.2:** Let $P'$ be a specification of $P$, then $s(f,P) \leq s(f,P') \leq S(f,P') \leq S(f,P)$; for arbitrary $P_1, P_2$, $s(f,P_1) \leq S(f,P_2)$

**D 5.3:** $f$ bounded is integrable if $s(f) = S(f)$ and the integral is $\int_a^b f(x) \, \mathrm{d}x$

**T 5.4:** $f$ bounded, integrable $\iff \forall \varepsilon > 0 \;\; \exists P \in \mathcal{P}(I)$ with $S(f,P) - s(f,P) \leq \varepsilon$ where $\mathcal{P}(I)$ is the set of all partitions of $I$

**T 5.9:** $f$ integrable $\iff \forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t. $\forall P \in \mathcal{P}_\delta(I), S(f,P) - s(f,P) < \varepsilon$, where $\mathcal{P}_\delta(I)$ is set of $P$ for which $\max_{1 \leq i \leq n} \delta_i \leq \delta$

**C 5.10:** $f$ integrable with $A := \int_a^b f(x) \, \mathrm{d}x \iff \forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t. $\forall P \in \mathcal{P}(I)$ with $\delta(P) < \delta$ and $\xi_1, \ldots, \xi_n$ with $\xi_i \in [xi - 1, x_i]$ and $P = \{x_0, \ldots, x_n\}$, $\left| A - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \varepsilon$

## 5.2 Integrable functions

**T 5.1:** $f, g$ bounded, integrable and $\lambda \in \mathbb{R}$. Then $f + g$, $\lambda \cdot f$, $f \cdot g$, $|f|$, $\max(f,g)$, $\min(f,g)$ and $\frac{f}{g}$(if $|g(x)| \geq \beta > 0 \;\; \forall x \in [a,b]$ are all integrable **C 5.3:** Let $P, Q$ be polynomials and $Q$ has no zeros on $[a,b]$. Then: $[a,b] \to \mathbb{R}$ and $x \mapsto \frac{P(x)}{Q(x)}$ integrable

**D 5.4:** *(uniform continuity)* if $\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ **T 5.6:** $f$ continuous on compact interval $I = [a,b] \implies f$ is uniformly continuous on $I$ **T 5.7:** $f$ continuous $\implies f$ integrable **T 5.8:** $f$ monotone $\implies f$ integrable

**T 5.10:** $I \subset \mathbb{R}$ compact interval with $I = [a,b]$ and $f_1, f_2$ bounded, integrable and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Then: $\int_a^b (\lambda_1 f_1(x) + \lambda_2 + f_2(x)) \, \mathrm{d}x = \lambda_1 \int_a^b f_1(x) \, \mathrm{d}x + \lambda_2 \int_a^b f_2(x) \, \mathrm{d}x$

## 5.3 Inequalities and Intermediate Value Theorem

**T 5.1:** $f, g$ bounded, integrable and $f(x) \leq g(x) \forall x \in [a,b]$, then $\int_a^b f(x) \, \mathrm{d}x \leq \int_a^b g(x) \, \mathrm{d}x$ **C 5.2:** if $f$ bounded, integrable, $\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x$ **T 5.3:** Let $f, g$ bounded, integrable, then $\left| \int_a^b f(x) g(x) \, \mathrm{d}x \right| \leq \sqrt{\int_a^b f^2(x) \, \mathrm{d}x} \cdot \sqrt{\int_a^b g^2(x) \, \mathrm{d}x}$

**T 5.4:** *(Intermediate Value Theorem)* $f$ continuous. Then $\exists \xi \in [a,b]$ s.t. $\int_a^b \, \mathrm{d}x = f(\xi)(b - a)$ **T 5.6:** Let $f$ continuous, $g$ bounded and integrable with $g(x) \geq 0 \;\; \forall x \in [a,b]$. Then $\exists \xi \in [a,b]$ s.t. $\int_a^b f(x) g(x) \, \mathrm{d}x = f(\xi) \int_a^b g(x) \, \mathrm{d}$

## 5.4 Fundamental theorem of Calculus

**First Fundamental Theorem of Calculus** — **Theorem 5.1**

Let $a < b$ and $f : [a,b] \to \mathbb{R}$ continuous. The function

$$F(x) = \int_a^x f(t) \, \mathrm{d}t, \quad a \leq x \leq b$$

is differentiable in $[a,b]$ and $F'(x) = f(x) \;\; \forall x \in [a,b]$

**Proof:** Split the integral: $\int_a^{x_0} f(t) \, \mathrm{d}t + \int_{x_0}^x f(t) \, \mathrm{d}t = \int_a^x f(t) \, \mathrm{d}t$, so $F(x) - F(x_0) = \int_{x_0}^x f(t) \, \mathrm{d}t$. Using the Intermediate Value Theorem, we get $\int_{x_0}^x f(t) \, \mathrm{d}t = f(\xi)(x - x_0)$ and for $x \neq x_0$ we have $\frac{F(x) - F(x_0)}{x - x_0} = f(\xi)$ and since $\xi$ is between $x_0$ and $x$ and since $f$ continuous, $\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$ □

**D 5.2:** *(Anti-derivative)* $F$ for $f$ if $F$ is differentiable in $[a,b]$ and $F' = f$ in $[a,b]$

**Second Fundamental Theorem of Calculus** — **Theorem 5.3**

$f$ as in 5.4.1. Then there exists an anti-derivative $F$ of $f$ that is uniquely determined bar the constant of integration and

$$\int_a^b f(x) \, \mathrm{d}x = F(a) - F(b)$$

**Proof:** Existence of $F$ given by 5.4.1. If $F_1$ and $F_2$ are anti-derivatives of $f$, then $F_1' - F_2' = f - f = 0$, i.e. $(F_1 - F_2)' = 0$. From 4.2.5 (1) we have that $F_1 - F_2$ is constant. We have $F(x) = C + \int_a^x f(t)\ dt$, where $C$ is an arbitrary constant. Especially, $F(b) = C + \int_a^b f(t)\ dt, F(a) = C$ and thus $F(b) - F(a) = C + \int_a^b f(t)\ dt - C = \int_a^b f(t)\ dt$

**T 5.5:** *(Integration by parts)* $\displaystyle\int_a^b f(x)g'(x)\ dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\ dx$. Be wary of cycles

**T 5.6:** *(Integration by substitution)* $\phi$ continuous and differentiable. Then $\displaystyle\int_a^b f(\phi(t))\phi'(t)\ dt = \int_{\phi(a)}^{\phi((b))} f(x)\ dx$

To use the above, in a function choose the inner function appropriately, differentiate it, substitute it back to get a more easily integrable function. **C 5.9:** $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ continuous

1. Let $a, b, c \in \mathbb{R}$ s.t. the closed interval with endpoints $a + c, b + c$ is contained in $I$. Then
$$\int_{a+c}^{b+c} f(x)\ dx = \int_a^b f(t + c)\ dt$$
2. Let $a, b, c \in \mathbb{R}, c \neq 0$ s.t. the closed interval with endpoints $ac, b$ is contained in $I$. Then
$$\frac{1}{c}\int_{ac}^{bc} f(x)\ dx = \int_a^b f(ct)\ dt$$

## 5.5 Integration of converging series

**T 5.1:** Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of bounded, integrable functions converging uniformly to $f$. Then $f$ bounded, integrable and $\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx$ **C 5.2:** $f_n$ s.t. the series converges. Then $\sum_{n=0}^{\infty} \int_a^b f_n(x)\ dx = \int_a^b \left(\sum_{n=0}^{\infty} f_n(x)\right)\ dx$
**C 5.3:** $f(x) = \sum_{n=0}^{\infty} x_k x^k$ with $\rho > 0$. Then $\forall 0 \leq r < \rho$, $f$ integrable on $[-r, r]$ and $\forall x \in ]-\rho, \rho[, \int_0^x f(t)\ dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$

## 5.6 Euler-McLaurin summation

**D 5.1:** $\forall k \geq 0$, the $k$-th Bernoulli-Polynomial $B_k(x) = k!P_k(x)$, where $P_k' = P_{k-1}\ \forall k \geq 1$ and $\int_0^1 P_k(x)\ dx = 0\ \forall k \geq 1$
**D 5.2:** Let $B_0 = 1$. $\forall k \geq 2\ B_{k-1}$ is given recursively by $\sum_{i=0}^{k-1} \binom{k}{i} B_i = 0$ **T 5.3:** *(McLaurin Series)* $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$
**T 5.5:** $f$ $k$ times continuously differentiable, $k \geq 1$. Then for $\widetilde{B_k}(x) = \begin{cases} B_k(x) & \text{for } 0 \leq x < 1 \\ B_k(x - n) & \text{for } n \leq x \leq n + 1 \text{ where } n \geq 1 \end{cases}$ that

1. For $k = 1$: $\sum_{i=1}^{n} f(i) = \int_0^n f(x)\ dx + \frac{1}{2}(f(n) - f(0)) + \int_0^n \widetilde{B_1}(x)f'(x)\ dx$ below: $\widetilde{R_k} = \frac{(-1)^{k-1}}{k!} \int_0^n \widetilde{B_k}(x)f^{(k)}(x)\ dx$
2. For $k \geq 2$: $\displaystyle\sum_{i=1}^{n} f(i) = \int_0^n f(x)\ dx + \frac{1}{2}(f(n) - f(0)) + \sum_{j=2}^{k} \frac{(-1)^j B_j}{j!}(f^{(j-1)}(n) - f^{(j-1)}(0)) + \widetilde{R_k}, \widetilde{R_k} = \sum_{(-1)^{(k-1)}}^{k!} \int_0^n \widetilde{B_1}(x)f^{(k)}(x)\ dx$

## 5.7 Stirling's Formula

**T 5.1:** $n! = \dfrac{\sqrt{2\pi n}n^n}{e^n} \cdot \exp\left(\dfrac{1}{12n} + R_3(n)\right)$, $|R_3(n)| \leq \frac{\sqrt{3}}{216} \cdot \frac{1}{n^2}\ \ \forall n \geq 1$ **L 5.2:** $\forall m \geq n + 1 \geq 1 : |R_3(m, n)| \leq \frac{\sqrt{3}}{216}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$

## 5.8 Improper Integrals

**D 5.1:** $f$ bounded and integrable on $[a, b]$. If $\displaystyle\lim_{b\to\infty} \int_a^b f(x)\ dx$ exists, we denote it $\int_a^{\infty} f(x)\ dx$ and call $f$ integrable on $[a, +\infty[$
**L 5.3:** $f : [a, \infty[ \to \mathbb{R}$ bounded and integrable on $[a, b] \forall b > 0$. If $|f(x) \leq g(x)\ \ \forall x \geq a$ and $g(x)$ integrable on $[a, \infty[$, then $f$ is integrable on $[a, \infty[$. If $0 \leq g(x) \leq f(x)$ and $\int_a^{\infty} g(x)\ dx$ diverges, so does $\int_a^{\infty} f(x)\ dx$ **T 5.5:** $f : [1, \infty[ \to [0, \infty[$ monotonically decreasing. $\sum_{n=1}^{\infty} f(n)$ converges $\Leftrightarrow \int_1^{\infty} f(x)\ dx$ converges **D 5.9:** If $f : ]a, b]$ is bounded and integrable on $[a + \varepsilon, b], \varepsilon > 0$, but not necessarily on $]a, b]$, then $f$ is integrable if $\lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)\ dx$ exists, then called $\int_a^b f(x)\ dx$
**D 5.12:** *(Gamma function)* For $s > 0$ we define $\Gamma(s) := \int_0^{\infty} e^{-x}x^{s-1}\ dx$
**T 5.13:** ***(1)*** $\Gamma(s)$ fulfills $\Gamma(1) = 1, \Gamma(s + 1) = s\Gamma(s)\ \ \forall s > 0$ and $\Gamma(\lambda x + (1 - \lambda)y) \leq \Gamma(x)^{\lambda}\Gamma(y)^{1-\lambda}\ \ \forall x, y > 0,\ \ \forall 0 \leq \lambda \leq 1$
***(2)*** $\Gamma(s)$ sole function $]0, \infty[\ \to\ ]0, \infty[$ that fulfills the above conditions. Additionally: $\Gamma(x) = \displaystyle\lim_{n\to\infty} \frac{n!n^x}{x(x + 1) \ldots (x + n)} \forall x > 0$
**T 5.14:** Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, for all $f, g : [a, b] \to \mathbb{R}$ continuous, we have $\int_a^b |f(x)g(x)|\ dx \leq ||f||_p||g||_q$

## 5.9 Partial fraction decomposition

Used for rational polynomial functions. Start by splitting the fraction into parts (usually factorized, so find zeros). Split denominator into the found parts, e.g. $\frac{a}{x-4} + \frac{b}{x+2}$, then expand to the same denominator on all fractions. Then $p(x)$ (the numerator) of the original fraction has to equal the new fraction's numerator, so use SLE to find coefficients. Get the numerator into the form of a polynomial, so e.g. $(a + b) \cdot x + (2a - 4b)$, then SLE is

$$\begin{vmatrix} 2 = a + b \\ -4 = 2a - b \end{vmatrix} \Leftrightarrow a = \frac{2}{3}, b = \frac{4}{3} \quad \text{for our rational polynomial } \frac{2x - 4}{x^2 - 2x - 8}$$

We can then insert our coefficients into the split fraction (here $\frac{a}{x-4} \ldots$) and we can integrate normally

# 6 Table of derivatives and Antiderivatives

| Antiderivative | Function | Derivative |
|---|---|---|
| $\frac{x^{n+1}}{n+1}$ | $x^n$ | $n \cdot x^{n-1}$ |
| $\ln\lvert x\rvert$ | $\frac{1}{x} = x^{-1}$ | $-x^{-2} = -\frac{1}{x^2}$ |
| $\frac{2}{3}x^{\frac{3}{2}}$ | $\sqrt{x} = x^{\frac{1}{2}}$ | $\frac{1}{2 \cdot \sqrt{x}}$ |
| $\frac{n}{n+1}x^{\frac{1}{n}+1}$ | $\sqrt[n]{x} = x^{\frac{1}{n}}$ | $\frac{1}{n}x^{\frac{1}{n}-1}$ |
| $e^x$ | $e^x$ | $e^x$ |
| $\exp(x)$ | $\exp(x)$ | $\exp(x)$ |
| $\frac{1}{a \cdot (n+1)}(ax+b)^{n+1}$ | $(ax+b)^n$ | $n \cdot (ax+b)^{n-1} \cdot a$ |
| $x \cdot (\ln\lvert x\rvert - 1)$ | $\ln(x)$ | $\frac{1}{x} = x^{-1}$ |
| $\frac{1}{\ln(a)} \cdot a^x$ | $a^x$ | $a^x \cdot \ln(a)$ |
| $\frac{x}{\ln(a)} \cdot (\ln\lvert x\rvert - 1)$ | $\log_a\lvert x\rvert$ | $\frac{1}{x \cdot \ln(a)}$ |
| $-\cos(x)$ | $\sin(x)$ | $\cos(x)$ |
| $\sin(x)$ | $\cos(x)$ | $-\sin(x)$ |
| $-\ln\lvert\cos(x)\rvert$ | $\tan(x)$ | $\frac{1}{\cos^2(x)}$ |
| $x \cdot \arcsin(x) + \sqrt{1-x^2}$ | $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $x \cdot \arccos(x) - \sqrt{1-x^2}$ | $\arccos(x)$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $x \cdot \arctan(x) - \frac{\ln(x^2+1)}{2}$ | $\arctan(x)$ | $\frac{1}{x^2+1}$ |
| $\ln\lvert\sin(x)\rvert$ | $\cot(x)$ | $-\frac{1}{\sin^2(x)}$ |
| $\cosh(x)$ | $\sinh(x)$ | $\cosh(x)$ |
| $\sinh(x)$ | $\cosh(x)$ | $\sinh(x)$ |
| $\ln\lvert\cosh(x)\rvert$ | $\tanh(x)$ | $\frac{1}{\cosh^2(x)}$ |
| | $\operatorname{arcsinh}(x)$ | $\frac{1}{\sqrt{1+x^2}}$ |
| | $\operatorname{arccosh}(x)$ | $\frac{1}{\sqrt{x^2-1}}$ |
| | $\operatorname{arctanh}(x)$ | $\frac{1}{1-x^2}$ |

**Logarithms**

*(Change of base)* $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ *(Powers)* $\log_a(x^y) = y\log_a(x)$
*(Div, Mul)* $\log_a(x \cdot (\div) y) = \log_a(x) + (-)\log_a(y)$
$\log_a(1) = 0 \ \ \forall a \in \mathbb{N}$

**Integration by parts** Should we get unavoidable cycle, where we have to integrate the same thing again, we may simply add the integral to both sides, and we thus have 2 times the integral on the left side and then finish the integration by parts on the right hand side and in the end divide by the factor up front to get the result.

**Inverse hyperbolic functions**

- $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$
- $\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$
- $\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

**Complement trick** $\sqrt{ax+b} - \sqrt{cx+d} = \frac{ax+b-(cx+d)}{\sqrt{ax+b}+\sqrt{cx+d}}$

**Values of trigonometric functions**

| ° | rad | $\sin(\xi)$ | $\cos(\xi)$ | $\tan(\xi)$ |
|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 1 |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{3}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | $\varnothing$ |
| 120° | $\frac{2\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ |
| 135° | $\frac{3\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-1$ |
| 150° | $\frac{5\pi}{6}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{2}$ |
| 180° | $\pi$ | 0 | $-1$ | 0 |

**Trigonometrie** $\cot(\xi) = \frac{\cos(\xi)}{\sin(\xi)}, \tan(\xi) = \frac{\sin(\xi)}{\cos(\xi)}$

$\sinh(x) := \frac{e^x - e^{-x}}{2} : \mathbb{R} \to \mathbb{R}$, $\cosh(x) := \frac{e^x + e^{-x}}{2} : \mathbb{R} \to [1, \infty]$, $\cosh(x) := \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} : \mathbb{R} \to [-1, 1]$

1. $\cos(x) = \cos(-x)$ and $\sin(-x) = -\sin(x)$
2. $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x)\sin(x)$
3. $\sin(x+w) = \sin(x)\cos(w) + \cos(x)\sin(w)$
4. $\cos(x+w) = \cos(x)\cos(w) - \sin(x)\sin(w)$
5. $\cos(x)^2 + \sin(x)^2 = 1$
6. $\sin(2x) = 2\sin(x)\cos(x)$
7. $\cos(2x) = \cos(x)^2 - \sin(x)^2$

**Further derivatives**

| $F(x)$ | $f(x)$ |
|---|---|
| $\frac{1}{a}\ln\lvert ax+b\rvert$ | $\frac{1}{ax+b}$ |
| $\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$ | $\frac{a(cx+d)-c(ax+b)}{(cx+d)^2}$ |
| $\frac{x}{2}f(x) + \frac{a^2}{2}\ln\lvert x+f(x)\rvert$ | $\sqrt{a^2+x^2}$ |
| $\frac{x}{2}f(x) - \frac{a^2}{2}\arcsin\left(\frac{x}{\lvert a\rvert}\right)$ | $\sqrt{a^2-x^2}$ |
| $\frac{x}{2}f(x) - \frac{a^2}{2}\ln\lvert x+f(x)\rvert$ | $\sqrt{x^2-a^2}$ |
| $\ln(x+\sqrt{x^2 \pm a^2})$ | $\frac{1}{\sqrt{x^2 \pm a^2}}$ |
| $\arcsin\left(\frac{x}{\lvert a\rvert}\right)$ | $\frac{1}{\sqrt{x^2-a^2}}$ |
| $\frac{1}{a}\arctan\left(\frac{x}{\lvert a\rvert}\right)$ | $\frac{1}{a^2-x^2}$ |

| $F(x)$ | $f(x)$ |
|---|---|
| $-\frac{1}{a}\cos(ax+b)$ | $\sin(ax+b)$ |
| $\frac{1}{a}\sin(ax+b)$ | $\cos(ax+b)$ |
| $x^x$ | $x^x \cdot (1+\ln\lvert x\rvert)$ |
| $(x^x)^x$ | $(x^x)^x \cdot (x + 2x\ln\lvert x\rvert)$ |
| $x^{(x^x)}$ | $x^{(x^x)} \cdot (x^{x-1} + \ln\lvert x\rvert \cdot x^x(1+\ln\lvert x\rvert))$ |
| $\frac{1}{2}(x - \frac{1}{2}\sin(2x))$ | $\sin(x)^2$ |
| $\frac{1}{2}(x + \frac{1}{2}\sin(2x))$ | $\cos(x)^2$ |