

# Autonomous Mobile Robots

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## 1 Introduction

### 1.1 Probability

**Def** (Sum rule)  $P(X) = \sum P(X, Y) = \sum P(X \cap Y)$

**Def** (Prod)  $P(X, Y) = P(X|Y)P(Y) = P(Y|X)P(X)$

**T** (Bayes)  $P(Y_i|X) = \frac{P(X|Y_i)P(Y_i)}{\sum_{j=1}^n P(X|Y_j)P(Y_j)}$

**Def** (Cont. Var) Sums become integrals  
e.g.  $\sum_X P(X) = 1$  becomes  $\int p(x) dx = 1$

**Def** (Indep.)  $x, y$  indep. iff  $p(x, y) = p(x)p(y)$

**Def** (Cond. Indep.) iff  $p(x, y|z) = p(x|z)p(y|z)$

**Def**  $E[x] = \int_{-\infty}^{\infty} xp(x) dx$ , also for  $x = f(x)$

**Def**  $\text{Cov}[x] = E[xx^T] - E[x]E[x]^T = \Sigma$

**Def** (Gauss. Dist.)  $x \sim \mathcal{N}(\mu, \Sigma)$  ( $\mu$  mean,  $\Sigma$  cov.),  
PDF:  $p(x) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu))$

### 1.2 Measurement models

$z = b_C + sM_S\omega + b + n + o$ :  $b_C$  const bias,  $b$  time bias,  $M$  missal.,  $n \sim \mathcal{N}(0, R)$  noise,  $s\omega$  corr. meas.,  $o$  other infl.

### 1.3 Trigonometry

## 2 Locomotion & Kinematics

### 2.1 Positioning

**Def** (Position Vector)  ${}^W t_B = {}^W t_{WB} B$ , Original Frame, End point, Target Frame,  $\sin = s$ ,  $\cos = c$

**Def** (State vector)  $x_R$ :  $x, v$  of rob in  $W$ , pos of sensors

**Def** (Rot. Mat.)  $R_z = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$   
 $R_y(\theta) = \begin{bmatrix} c(\psi) & 0 & s(\psi) \\ 0 & 1 & 0 \\ -s(\psi) & 0 & c(\psi) \end{bmatrix} R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\psi) & -s(\psi) \\ 0 & s(\psi) & c(\psi) \end{bmatrix}$

**R** Application:  ${}^W a = R_{WB} B a$

**L**  $R_{BW} = R_{WB}^{-1} = R_{WB}^T$ ,  $\det(R_{WB}) = 1$  (orth.)

**R** Cols of  $R_{WB}$  are basis vec. of Frame  $\vec{F}_B$  in  $\vec{F}_W$

**Def** (Euler Angles) Yaw ( $z$ ), Pitch ( $y$ ), Roll ( $x$ ), mult. rotation matrices, e.g.  $R_{EB} = R_z(\psi) \cdot R_y(\theta) \cdot R_x(\varphi)$ , **bound.**  $[n]^\times = n x^T$  (matrix from vec + arg  $x$ )

**Def** (Rot. Vec)  $\alpha = \alpha n$  ( $n$  normal)

$R(\alpha, n) = I_3 + \sin(\alpha)[n]^\times + (1 - \cos(\alpha))([n]^\times)^2$

**Def** (Quaternions)  $q = q_w + q_x i + q_y j + q_z k$  with  $i^2 = j^2 = k^2 = -1$ ,  $(ij = -ji = k, \text{ same for } jk \text{ and } ki)$

**Def** (Transf. M)  $T_{AB} = \begin{bmatrix} R_{AB} & A t_B \\ 0_{1 \times 3} & 1 \end{bmatrix}$

$T_{BA} = T_{AB}^{-1} = \begin{bmatrix} R_{AB}^T & -R_{AB}^T A t_B \\ 0_{1 \times 3} & 1 \end{bmatrix}$   $T_{AC} = T_{AB} T_{BC}$

### 2.2 Forward Kinematics (FK)

$T_{WB_n}(\theta) = T_{WB_0} T_{B_0 B_1}(\theta_1) \cdots T_{B_{n-1} B_n}(\theta_n)$ .

For 2R system:  ${}^W t_{WE} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$

With workspace (pos)  $W$  for  $\theta_1, \theta_2 \in [-\pi, \pi]$

### 2.3 Inverse Kinematics (IK)

**Option:** Solve Forward Kinematics for angles.

**Better:** Law of cosine with polar coordinates. Compute angle using cosine rule,

$\theta_1 = \phi \pm \alpha$ ,  $\theta_2 = \pm(\pi - \beta)$

(Positive for Elbow Down, Negative for Elbow Up)

**Extension to 6R:** 1. Waist: spherical coords (2 sol.)

2. 2 sols from 2R for shoulder + elbow

3. Solve for wrist joints (no influence on pos)

### 2.4 Temporal Models

For **Cont-time n-lin. system of ODE**  $\dot{x} = f_C(x(t), u(t))$ , with measurements  $z(t) = h(x(t)) + v(t)$ .

Need linearised (around  $f_C(\bar{x}, \bar{u}) = 0$ , at **equilibrium**):

$\delta \dot{x}(t) = f_C(\bar{x}, \bar{u}) + F_C \delta x(t) + G_C \delta u(t) + L_C w(t)$

$\delta z(t) = H \delta x(t) + v(t)$ . Herein,  $H$  is measurements,  $F_C$  system,  $G$  input gain,  $w$  process noise,  $v$  measurement noise, both zero-mean **Gaussian White Noise Process**.

For **n-lin. cont-time system**:  $\dot{x}(t) = f_C(x(t), u(t), w(t))$

$z(t) = h(x(t)) = v(t)$ , linearization is the same

To **discretize**, integrate from  $t_{k-1}$  to  $t_k$ :

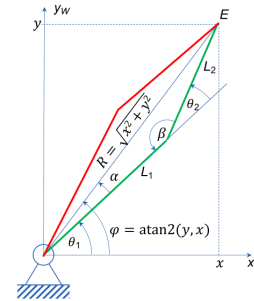
$x_k = f(x_{k-1}, u_k, w_k)$   $z_k = h(x_k) + v_k$ , **linearised**:

$\delta x_k = f(\bar{x}, \bar{u}) + F \delta x_{k-1} + G \delta u_k + L w_k$ ;  $\delta z_k = H \delta x_k$

**Trapezoidal num. int**  $\Delta x_1 = \Delta t f_C(x_{k-1}, u_{k-1}, t_{k-1})$

$\Delta x_2 = \Delta t f_C(x_{k-1} + \Delta x_1, u_k, t_k)$ , then:

$x_k = x_{k-1} + 0.5 \cdot (\Delta x_1 + \Delta x_2)$



## 2.5 Rigid body & IMU kinematics

**Velocity**  ${}^I v_{IB} = \frac{d}{dt}({}^I t_B)$

**Rot. Velocity**  ${}^I \omega_{IB} = \frac{d}{dt}(\alpha) I t$

**Velocity point**  ${}^B v_{IP} = {}^B v_{IB} + {}^B \omega_{IB} \times B t_P$

**Rotation Matrices**

- For left perturbing  $\dot{R}_{IB} = [{}^I \omega_{IB}]^\times R_{IB}$
- For right perturbing  $\dot{R}_{IB} = R_{IB} [{}^I \omega_{IB}]^\times$
- Constant angular velocity ( $\exp[\Delta \alpha]^\times = \delta R(\Delta \alpha)$ )  
 $R_{IB}(t + \Delta t) = \exp[\Delta \alpha]^\times R_{IB}(t)$

**Quaternions**

- For left perturbing  $\dot{q}_{IB} = \frac{1}{2} \begin{bmatrix} {}^I \omega_{IB} \\ 0 \end{bmatrix} \otimes q_{IB}$
- For right perturbing  $\dot{q}_{IB} = \frac{1}{2} q_{IB} \otimes \begin{bmatrix} {}^B \omega_{IB} \\ 0 \end{bmatrix}$

**IMU** (Outputs  $s \tilde{a}$  (accel.),  $s \tilde{\omega}$  (rot. accel.))

${}^W t_S = {}^W v$ ,  $\dot{q}_{WS} = \frac{1}{2} q_{WS} \otimes \begin{bmatrix} s \tilde{\omega} + w_g - b_g \\ 0 \end{bmatrix}$

${}^W \dot{v} = R_{WS} (s \tilde{a} + w_a - b_a) + {}^W g$  where gray parts only IRL (in theor. models, leave out), with  $\dot{b}_g = w_{b_g}$  and  $\dot{b}_a = w_{b_a}$

**IMU Sensor Model:**  $\tilde{z} = b_C + s M z + b + n + o$  where bias  $b$  and scale  $s$  often modelled time-varying  $\dot{b}(t) = \sigma_C n(t)$ .  $b_C$  const. calib;  $M$  Misalignment;  $n$  noise;  $o$  other infl.

### 2.6 Rigid Body Dynamics

**Def** (Newton II) For fin. body w/ mass  $m$  and inertia mat.  $I$ , with force  $F$  and torque  $T$  on **Centre of Mass** (CoM), expressed in body frame:

$${}^B F = \sum {}^B F_i = m({}^B \dot{v}_{CoM}) + m {}^B \omega \times {}^B v_{CoM}$$

$${}^B T = \sum {}^B T_i = I({}^B \dot{\omega}) + {}^B \omega \times I {}^B \omega$$

${}^B v_{CoM}$  vel. of CoM,  ${}^B \omega$  rot. speed; both w.r.t. world frame

### 2.7 Wheeled robot Kinematics

**Non-holonomic systems not integrable**,

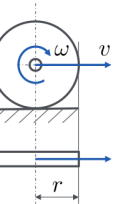
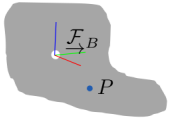
no inst. move in every direct.

**Wheel constraints**  $v_i = \omega_i r_i$

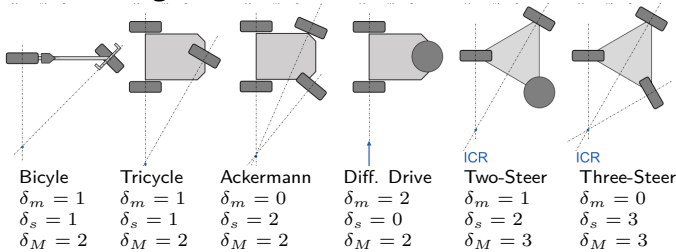
- Driving straight all  $v$  equal
- Turning Wheel axis must intersect the **Instant Centre of Rotation** (ICR), speeds:  $v_i \div R_i = \Omega$  ( $R_i$  dist. wheel-ICR,  $\Omega$ , vehicle body rotation rate)

**Maneuverability**

- Deg. of Mobility:  $\delta_m = 3 - \#$ constrained directions
- Deg. of Steerability:  $\delta_s = \#$ steerable wheels
- Deg. of Maneuverability:  $\delta_M = \delta_m + \delta_s$



## Wheel Configurations



## Differential Drive Kinematics

**State vec**  $x = [x_1, x_2, \theta]^\top$ , **Inputs**  $u = [\omega_l, \omega_r]^\top$ ,  $r_r$  radius of right wheel,  $w$  width of robot

**Gen. eq. of Motion**  $\dot{x}_1 = v \cos(\theta)$ ,  $\dot{x}_2 = v \sin(\theta)$ ,  $\dot{\theta} = \Omega$ , with  $v = 0.5 \cdot (\omega_l r_l + \omega_r r_r)$ ,  $\Omega = \frac{\omega_r r_r - \omega_l r_l}{w}$

**Straight:**  $v = \omega_l r_l = \omega_r r_r$ ,  $\Omega = 0$ ,  $D = v \Delta t$ .

$$b_s = \begin{bmatrix} D \cos(\theta) \\ D \sin(\theta) \\ 0 \end{bmatrix} \quad b_t = \begin{bmatrix} R(\sin(\Delta\theta + \theta) - \sin(\theta)) \\ -R(\cos(\Delta\theta + \theta) - \cos(\theta)) \\ \Delta\theta \end{bmatrix}$$

**Turning:**  $\Omega = (\omega_l r_l) / R_l = (\omega_r r_r) / R_r$ ,  $R = v / \Omega$ ,  $\Delta\theta = \Omega \Delta t$

**Discretized:**  $x_k = x_{k-1} b_i$  with  $i \in \{s, t\}$ . ( $\int \dots d\Delta t$ )

## 3 Sensors & Actuators

**Meas. Model:**  $z = h(x) + v + o$ , with  $h(x)$  deterministic mean,  $v$  zero-mean noise,  $o$  unmodelled effects,  $x$  true state

**Motor encoders** Typ. 64-2048 incrm. per rev; Estim. rot  
**Rolling-Shutter** Most CMOS sensors don't take full image at once, need time stamp for each row

### 3.1 GNSS

Need ultra-precise time sync ( $c \approx 0.3 \text{ m/ns}$ ). **Errors**

- Multipath problem (signal bounce) (0.5 - 100m)
- Ionosphere delays (10m)
- Satellite pos. err, trop. delay (1m)

### 3.2 Actuators

**Hydraulic** acc., easy control, power; maint., speed, price

**Pneumatic** price, shock abs., speed; acc., loud, maint.

#### 3.2.1 DC Motor

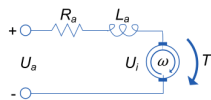
(Kirchoff)  $U_a = L_a \dot{I}_a + R_a I_a + U_i$

(Torque, Lorenz Force)  $T = k_T I_a$

(Induced V, Faraday)  $U_i = k_i \omega$

(Mech. pow. eq. el. pow)

$U_i I_a = k_i \omega I_a = T \omega \Rightarrow k_i = k_T =: k$



### 3.3 Cameras

**Def** (Pinhole projection)  $\begin{bmatrix} u \\ v \end{bmatrix}^\top = \frac{f}{z} \begin{bmatrix} x \\ y \end{bmatrix}^\top$  with  $f$  the distance to the lens and  $z$  the full distance

$u = c_u + f \cdot x'$  and  $v = c_v + f \cdot y'$  where  $x' = t_x \div t_z$  and  $y' = t_y \div t_z$  where  $u, v$  are the pixel  $x, y$  coords,  $c = [c_u, c_v]^\top$  is optical centre of cam in pixel coords,  $f$  scale factor, and  $c t_P = [t_x, t_y, t_z]^\top$

$$\text{The full proj: } \mathbf{u} = \begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} f & 0 & c_u \\ 0 & f & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \mathbf{K} c t_P$$

If  $p$ . in diff frame  $w t_P$ , then  $\mathbf{u} = \mathbf{K} [\mathbf{R}_{CW} c t_{CW}] = w t_P$

#### 3.3.1 Pinhole Camera Projection with distortion

**Def** Model:  $\mathbf{u} = \mathbf{k}(d(p(c t_P)))$ , with:

(Projection)  $\mathbf{x}' = p(c t_P) = t_z^{-1} \cdot [t_x, t_y]^\top$

(Distortion model,  $r^2 = x'^2 + y'^2$ ,  $\mathbf{x}'' = d(\mathbf{x}')$ )

$$\mathbf{x}'' = \frac{1+k_1 r^2+k_2 r^4+k_3 r^6}{1+k_4 r^2+k_5 r^4+k_6 r^6} \mathbf{x}' + \begin{bmatrix} 2p_1 x' y' + p_2 (r^2 + 2x'^2) \\ p_1 (r^2 + 2y'^2) + 2p_2 x' y' \end{bmatrix}$$

(Scale and Centre)  $\mathbf{u} = \mathbf{k}(\mathbf{x}'') = \text{diag}([f_u, f_v]) \cdot \mathbf{x}'' + \mathbf{x}$

All with  $k_i$  radial distortion params, optional for  $i > 2$ ,  $p_i$

tang. dist. param,  $f_u, f_v$  focal length in pixels

**Inverse**  $c r = [d^{-1}(k^{-1}(\mathbf{u})), 1]^\top$

(To unit plane)  $\mathbf{x}'' = k^{-1}(\mathbf{u}) = [f_u^{-1}, f_v^{-1}]^\top (\mathbf{u} - \mathbf{c})$

(Un-distort)  $\mathbf{x}' = d^{-1}(\mathbf{x}'')$  (usually comp. numerically)

(Compute ray)  $c r = [\mathbf{x}', 1]^\top$

#### 3.3.2 Undistorting a whole image

$\mathbf{u}_i = \mathbf{k}(d(k_{\text{new}}^{-1}(\mathbf{u}_{i,\text{new}})))$  where  $\mathbf{u}_{i,\text{new}}$  is the place of pixel in output,  $\mathbf{u}_i$  is the input

**Omnidir. Cam** undistortion model with  $f(u, v) = \sum_{i=0}^N a_i \rho^i$  with  $\rho = \sqrt{(u - c_u)^2 + (v - c_v)^2}$ ,  $N = 4$  accurately describes it for most fisheye and catadioptric cameras

## 3.4 Depth and Range sensing

### 3.4.1 Triangulation-based

**Struct. Light** Single cam, single projector: Spatial acc, no worky in bright light, interference with other IR depth cams

**Active Stereo** 2 cams, 1 proj: worky in bright light, need stereo matching, less accurate, error grows with distance

### 3.4.2 Classic Stereo

Both images: same plane, focal length, centre,  $x$ -axis. Given corresponding pixels  $[u_l, v]$  and  $[u_r, v]$ ,  $z = \frac{b \cdot f}{u_r - u_l}$  with  $u_l = f \cdot \frac{x}{z} + c_u$  and  $u_r = f \cdot \frac{x-b}{z} + c_u$

### 3.4.3 Time of Flight, Projection

No occlusions/shadows, Interference with other dev, multi-path leading to larger distances sensed

$$\text{Proj. } z = \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} \mathbf{k}(d(p(c t_P))) \\ [0, 0, 1]_c t_P \end{bmatrix} \quad \text{Back: } c t_P = \begin{bmatrix} d x' \\ d \end{bmatrix}$$

### 3.4.4 Range Sensors

**Ultrasonic** Typ. freq: 40kHz - 180kHz, Range: 12cm - 5m, Acc:  $\approx 2\text{cm}$ , rel error  $\approx 2\%$  meas. for transp. surf., cheap(ish), Cone wider, reflect. angle dep, wind / currents

**LiDAR** Time-of-Flight-based, Accuracy, range, works in bright light, Complex, expensive, one timestamp per measurement.

Typical Ranges: up to 100m

## 4 Multi-Sensor Estimation

### 4.1 Linearization

$f(\mathbf{x}) \approx f(\bar{\mathbf{x}}) + \mathbf{J}_f|_{\mathbf{x}=\bar{\mathbf{x}}} (\mathbf{x} - \bar{\mathbf{x}})$ ,  $f'$ , no vec in 1D;  $\bar{\mathbf{x}}$  lin. p.

**Def** (Jac.)  $\mathbf{J}_f$  rows for eq of  $\mathbf{f}$  cols for vars of each eq.

Approximation using finite differences  $\frac{f(\bar{\mathbf{x}}+h) - f(\bar{\mathbf{x}})}{h}$ ,

or central differences (vector of  $\frac{f(\bar{\mathbf{x}}+h_i \mathbf{e}_i) - f(\bar{\mathbf{x}})}{h_i}$ , with  $\mathbf{e}_i$  unit vec)

### 4.2 Linear Least Squares

**Goal:**  $\text{argmin}_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$ ,  $\mathbf{A}$ : rows  $i$ -th datap. col  $c$ :  $t_i^{c-1}$ .

**Man. sol.:** comp.  $M = \mathbf{A}^\top \mathbf{A}$ ,  $\mathbf{b}' = \mathbf{A}^\top \mathbf{b}$ , then  $M\mathbf{x} = \mathbf{b}'$ .

**Prob. sol.:**  $\text{argmax}_p(\mathbf{x} | z)$  with

• **Max. Like**  $p(\mathbf{z} | \mathbf{x}) \propto p(\mathbf{z} | \mathbf{x}) = \prod_{i=1}^N p(z_i | \mathbf{x})$

• **M a Post**  $p(\mathbf{x} | \mathbf{z}) \propto p(\mathbf{z} | \mathbf{x}) p(\mathbf{x}) = p(\mathbf{x}) \prod_{i=1}^N p(z_i | \mathbf{x})$

### 4.3 Non-Linear Least Squares

Find  $\mathbf{x}^* = \text{argmax}_p(\mathbf{x} | z) = \text{argmin}(-\log(p(\mathbf{x} | z)))$

**Gauss-Newton**

**Levenberg-Marquardt**