

1 Linear Algebra

Relevant definitions used throughout Analysis II.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

Def Scalar Product $x \cdot y := \sum_{i=1}^n (x_i \cdot y_i)$

Def Euclidian Norm $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$

Used to generalize $|x|$ in many Analysis I definitions

Lem. Properties of $\|x\|$

- (i) $\|x\| \geq 0$
- (ii) $\|x\| \iff x = 0$
- (iii) $\|\alpha x\| = \alpha \cdot \|x\|$
- (iv) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

Def Definiteness

Positive Definite $\stackrel{\text{def}}{\iff} x^\top \mathbf{A} x > 0 \quad \forall x \in \mathbb{R}_{\neq 0}^n$

Negative Definite $\stackrel{\text{def}}{\iff} x^\top \mathbf{A} x < 0 \quad \forall x \in \mathbb{R}_{\neq 0}^n$

If 0 is allowed, \mathbf{A} is called positive/negative semi-definite.

Def Trace $\text{Tr}(\mathbf{A}) := \sum_{i=1}^{\min(m,n)} \mathbf{A}_{i,i}$

2 Differential Equations

Def Differential Equation (DE)

Equation relating unknown f to derivatives $f^{(i)}$ at *same* x .

Def Ordinary Differential Equation (ODE)

DE s.t. $f : I \rightarrow \mathbb{R}$ is in one variable.

Def Partial Differential Equation (PDE)

DE s.t. $f : I^d \rightarrow \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def Order $\text{ord}(F) := \max_{i \geq 0} \{i \mid f^{(i)} \in F, f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \rightarrow \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$$\exists f : I \rightarrow \mathbb{R}$$

$$\text{s.t. } \forall x \in I : f'(x) = F(x, f(x)) \text{ and } f(x_0) = y_0$$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for *nice enough* equations.

2.1 Linear Differential Equations

Def Linear Differential Equation (LDE)

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

$I \subset \mathbb{R}$ is open, $k \geq 1$, $\forall i < k : a_i : I \rightarrow \mathbb{C}$

Def Homogeneity of LDEs

Homogeneous $\stackrel{\text{def}}{\iff} b = 0$

Inhomogeneous $\stackrel{\text{def}}{\iff} b \neq 0$

Remark $D(y) := y^{(k)} + \dots + a_0y$ is a linear operation:

$$D(z_1f_1 + z_2f_2) = z_1D(f_1) + z_2D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$ k -times differentiable

Def Homogeneous Solution Space

$$\mathcal{S}(F) := \{f : I \rightarrow \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1D(f_1) + z_2D(f_2) = 0$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$

Solutions for complex homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and complex-valued

1. \mathcal{S} is a complex vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and real-valued

1. \mathcal{S} is a real vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def Inhomogeneous Solution Space

$$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), f_0 \text{ is a particular sol.}\}$$

Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous, $b : I \rightarrow \mathbb{C}$

1. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark Applications of Linearity

If f_1 solves F for b_1 , and f_2 for b_2 : $f_1 + f_2$ solves $b_1 + b_2$.

Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

3 Solutions to Differential Equations

3.1 Linear Solutions: First Order

Form: $y' + ay = b \quad I \subset \mathbb{R}, \quad a, b : I \rightarrow \mathbb{R}$

Approach:

- 1. Hom. Solution f_1 for: $y' + ay = 0$
Note that \mathcal{S} has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for \mathcal{S}
- 2. Part. Solution f_0 for $y' + ay = b$

Solutions: $f_0 + z f_1 \quad \text{for } z \in \mathbb{C}$

Explicit Homogeneous Solution

$A(x)$ is a primitive of a , $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$
$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

Method **Variation of Constants:** Treating z as $z(x)$ yields:

Explicit Inhomogeneous Solution

$A(x)$ is a primitive of a

$$f_0(x) = \underbrace{\left(\int b(x) \cdot \exp(A(x)) \right)}_{z(x)} \cdot \exp(-A(x))$$

Method **Educated Guess**
Usually, y has a similar form to b :

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \cdot \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cdot \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$a e^{\alpha x} \cdot \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$b e^{\alpha x} \cdot \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

Remark If α, β are roots of $P(X)$ with multiplicity j , multiply guess with a $P_j(x)$.

3.2 Linear Solutions: Constant Coefficients

Form:

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Where $a_0, \dots, a_{k-1} \in \mathbb{C}$ are constants, $b(x)$ is continuous.

3.2.1 Homogeneous Equations

The idea is to find a Basis of \mathcal{S} :

Def **Characteristic Polynomial** $P(X) = \prod_{i=1}^k (X - \alpha_i)$

Remark The unique roots $\alpha_1, \dots, \alpha_l$ form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}$$

v_1, \dots, v_k are the Multiplicities of $\alpha_1, \dots, \alpha_k$

Remark If $\alpha_j = \beta + \gamma i \in \mathbb{C}$ is a root, $\bar{\alpha}_j = \beta - \gamma i$ is too. To get a real-valued solution, apply:

$$e^{\alpha_j x} + e^{\bar{\alpha}_j x} = e^{\beta x} (\cos(\gamma x) + \sin(\gamma x))$$

Explicit Homogeneous Solution

Using $\alpha_1, \dots, \alpha_k$ from $P(X)$ s.t. $\alpha_i \neq \alpha_j, z_i \in \mathbb{C}$ arbitrary

$$f(x) = \prod_{i=1}^k z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)}(x) = \prod_{i=1}^k z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of \mathcal{S}

Solutions exist $\forall Z = (z_1, \dots, z_k)$ since that system's $\det(M_Z) \neq 0$.

3.2.2 Inhomogeneous Equations

Method **Undetermined Coefficients:** An educated guess.

- 1. $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$
 $\deg(Q) \leq d + v_\alpha$, where v_α is α 's multiplicity in $P(X)$
- 2. $\left. \begin{aligned} b(x) &= cx^d \cdot \cos(\alpha x) \\ b(x) &= cx^d \cdot \sin(\alpha x) \end{aligned} \right\} f_p = Q_1(x) \cos(\alpha x) + Q_2(x) \sin(\alpha x)$
 $\deg(Q_{1,2}) \leq d + v_\alpha$, where v_α is α 's multiplicity in $P(X)$

Remark **Applying Linearity**

If $b(x) = \sum_{i=1}^n b_i(x)$, A solution for $b(x)$ is $f(x) = \sum_{i=1}^n f_i(x)$
Sometimes called *Superposition Principle* in this context

3.3 Other Methods

Method **Change of Variable**

If $f(x)$ is replaced by $h(y) = f(g(y))$, then h is a sol. too.
Changes like $h(t) = f(e^t)$ may lead to useful properties.

Separation of Variables

Form:

$$y' = a(y) \cdot b(x)$$

Solve using:

$$\int \frac{1}{a(y)} dy = \int b(x) dx + c$$

Usually $\int 1/a(y) dy$ can be solved directly for $\ln |a(y)| + c$.

3.4 Method Overview

Method	Use case
Variation of constants	LDE with $\text{ord}(F) = 1$
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. $y' = a(y) \cdot b(x)$
Change of Variables	e.g. $y' = f(ax + by + c)$

4 Continuous functions in \mathbb{R}^n

Treating functions $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{C}/\mathbb{R}^m$, $m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ means:
 $x = (x_1, \dots, x_n)$, $f(x) = (f_1(x), \dots, f_m(x))$

4.1 Multivariate functions

Def Linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

In other words: $f(x) = Ax$, $A \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

Def Affine Linear map $f(x) \mapsto Ax + c$

Def Quadratic form $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

In other words: $Q(x) = \sum_{i=0}^n \sum_{j=0}^m (a_{i,j} x_i x_j)$

Def Monomials $M(x) : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto \alpha x_1^{d_1} \dots x_n^{d_n}$

For example: $f(x, y, z) = 16x^2 y z^5$

Def $\deg(M) := e = \sum_{i=1}^n d_i$

For example: $\deg(16x^2 y z^5) = 8$

Def Polynomials $P(x) := \sum_{i=0}^n M_i(x)$

For example: $P(x, y, z) = x^3 + 25x^2 y^6 z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$

For example: $\deg(x^3 + 25x^2 y^6 z + xy) = 9$

Visualisations for some function types:

Def Graph $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$

Only for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Visually, this is a surface in \mathbb{R}^3

Def Vector Plots for $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Points in $(x, y) \in \mathbb{R}^2$ are displayed as vectors $f(x, y)$

4.2 Sequences in \mathbb{R}^n

Def Sequences in \mathbb{R}^n

$(x_k)_{k \geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \dots, x_{k,n})$

Def Convergence in \mathbb{R}^n

$$\lim_{k \rightarrow \infty} (x_k) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

Lem. Equivalent conditions to Convergence

$$(i) \quad \forall i \text{ s.t. } 1 \leq i \leq n : \lim_{k \rightarrow \infty} (x_{k,i}) = y_i$$

$$(ii) \quad \lim_{k \rightarrow \infty} \|x_k - y\| = 0$$

Def Limits at points

$$\lim_{x \neq x_0 \rightarrow x_0} (f(x)) = y \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 : \\ \forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m, \quad x_0 \in X, \quad y \in \mathbb{R}^m$$

The sequence test for Continuity works for point-limits too.

4.3 Continuity in \mathbb{R}^n

Def Continuity in \mathbb{R}^n

f continuous at $x_0 \in X \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

$$f \text{ continuous} \stackrel{\text{def}}{\iff} \forall x \in X : f \text{ continuous at } x$$

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m$$

Lem. Continuity using Sequences

f continuous at x_0 if and only if:

$$\forall (x_k)_{k \geq 1} : \lim_{k \rightarrow \infty} (x_k) = x_0 \implies \lim_{k \rightarrow \infty} (f(x_k)) = f(x_0)$$

$$X \subset \mathbb{R}^n, \quad f : X \rightarrow \mathbb{R}^m$$

Lem. Continuity of Compositions

$f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p$ continuous $\implies g \circ f$ continuous

$$X \subset \mathbb{R}^n, \quad Y \subset \mathbb{R}^m, \quad p \geq 1$$

Lem. Continuity using Coordinate Functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

4.4 Subsets of \mathbb{R}^n

Def Bounded

$X \subset \mathbb{R}^n$ bounded $\stackrel{\text{def}}{\iff} \left\{ \|x\| \mid x \in X \right\} \subset \mathbb{R}$ bounded.

Example: The open disc $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is bounded.

Def Closed

$X \subset \mathbb{R}^n$ closed $\stackrel{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \lim_{k \rightarrow \infty} (x_k) \in X$

Example: \emptyset, \mathbb{R}^n are closed.

Def Compact if closed and bounded.

Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is compact.

Def Open

$X \subset \mathbb{R}^n$ open $\stackrel{\text{def}}{\iff} \forall x \in X, \exists \delta > 0 :$

$$\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall i \leq n\} \subset X$$

In other words: Changing any coord. x_i by δ keeps x' in X

Example: \emptyset, \mathbb{R}^n are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. Continuous functions preserve closed/open

\forall closed/open $Y :$

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous, $Y \subset \mathbb{R}^m$

Lem. The complement of open sets is closed

$$X \subset \mathbb{R}^n \text{ is open} \iff \underbrace{\{x \in \mathbb{R}^n \mid x \notin X\}}_{\text{Complement}} \text{ is closed}$$

Min-Max Theorem

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \rightarrow \mathbb{R}$:

$$\exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), \quad f(x_2) = \inf_{x \in X} f(x)$$

5 Differential Calculus in \mathbb{R}^n

5.1 Partial Derivatives

Partial Derivative

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $1 \leq i \leq n$, $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \rightarrow \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,1}, \dots, x_{0,t-1}, t, x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \rightarrow t}$$

Notation $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

Lem. Properties of Partial Derivatives

Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

- (i) $\partial_{x_i}(f + g) = \partial_{x_i} f + \partial_{x_i} g$
- (ii) $\partial_{x_i}(fg) = \partial_{x_i}(f)g + \partial_{x_i}(g)f$ if $m = 1$
- (iii) $\partial_{x_i}\left(\frac{f}{g}\right) = \frac{\partial_{x_i}(f)g - \partial_{x_i}(g)f}{g^2}$ if $g(x) \neq 0 \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f, g : X \rightarrow \mathbb{R}^n$, $1 \leq i \leq n$

The Jacobian

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \cdots & \partial_{x_n} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(x) & \partial_{x_2} f_m(x) & \cdots & \partial_{x_n} f_m(x) \end{bmatrix}$$

Think of f as a vector of f_i , then \mathbf{J}_f is that vector stretched for all x_j

Def Gradient $\nabla f(x_0) := \begin{bmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, i.e. must map to 1 dimension

Remark ∇f points in the direction of greatest increase.

This generalizes that in \mathbb{R} , $\text{sgn}(f)$ shows if f increases/decreases

Def Divergence $\text{div}(f)(x_0) := \text{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$, \mathbf{J}_f exists

5.2 The Differential

Partial derivatives don't provide a good approx. of f , unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in \mathbb{R}^n .

Differentiability in \mathbb{R}^n & the Differential

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$, $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If f is differentiable at $x_0 \in X$ with u s.t.

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{1}{\|x - x_0\|} \left(f(x) - f(x_0) - u(x - x_0) \right) = 0$$

Similarly, f is differentiable if this holds for all $x \in X$

Lem. Properties of Differentiable Functions

- (i) Continuous on X
- (ii) $\forall i \leq m, j \leq n : \partial_{x_j} f_i$ exists
- (iii) $m = 1 : \partial_{x_i} f(x_0) = a_i$
for: $u(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ differentiable on X

Lem. Preservation of Differentiability

- (i) $f + g$ is differentiable: $d(f + g) = df + dg$
- (ii) fg is differentiable, if $m = 1$
- (iii) $\frac{f}{g}$ is differentiable, if $m = 1$, $g(x) \neq 0 \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f, g : X \rightarrow \mathbb{R}^m$ differentiable on X

Lem. Cont. Partial Derivatives imply Differentiability

if all $\partial_{x_j} f_i$ exist and are continuous:

$$f \text{ differentiable on } X, \quad df(x_0) = \mathbf{J}_f(x_0)$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$

Lem. Chain Rule $g \circ f$ is differentiable on X

$$\begin{aligned} d(g \circ f)(x_0) &= dg(f(x_0)) \circ df(x_0) \\ \mathbf{J}_{g \circ f}(x_0) &= \mathbf{J}_g(f(x_0)) \cdot \mathbf{J}_f(x_0) \end{aligned}$$

$X \subset \mathbb{R}^n$ open, $Y \subset \mathbb{R}^m$ open, $f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p, f, g$ diff.-able

Def Tangent Space

$$T_f(x_0) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0) \right\}$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ diff.-able, $x_0 \in X$, $u = df(x_0)$

Def Directional Derivative

$$D_v f(x_0) = \lim_{t \neq 0 \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$, $v \neq 0 \in \mathbb{R}^n$, $x_0 \in X$

Lem. Directional Derivatives for Diff.-able Functions

$$D_v f(x_0) = df(x_0)(v) = \mathbf{J}_f(x_0) \cdot v$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ diff.-able, $v \neq 0 \in \mathbb{R}^n$, $x_0 \in X$

Remark $D_v f$ is linear w.r.t v , so: $D_{v_1+v_2} f = D_{v_1} f + D_{v_2} f$

Remark $D_v f(x_0) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$

In the case $f : X \rightarrow \mathbb{R}$, where θ is the angle between v and $\nabla f(x_0)$

5.3 Higher Derivatives

Def Differentiability Classes

$$\begin{aligned} f \in C^1(X; \mathbb{R}^m) &\stackrel{\text{def}}{\iff} f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \text{ exist} \\ f \in C^k(X; \mathbb{R}^m) &\stackrel{\text{def}}{\iff} f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \in C^{k-1} \\ f \in C^\infty(X; \mathbb{R}^m) &\stackrel{\text{def}}{\iff} f \in C^k(X; \mathbb{R}^m) \forall k \geq 1 \end{aligned}$$

$$X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}^m$$

Lem. Polynomials, Trig. functions and exp are in C^∞

Lem. Operations preserve Differentiability Classes

$$\begin{aligned} (i) \quad f + g &\in C^k \\ (ii) \quad fg &\in C^k \quad \text{if } m = 1 \\ (iii) \quad \frac{f}{g} &\in C^k \quad \text{if } m = 1, g(x) \neq 0 \forall x \in X \\ f, g &\in C^k \end{aligned}$$

Lem. Composition preserves Differentiability Classes

$$g \circ f \in C^k$$

$$f \in C^k, \quad f(X) \subset Y, \quad Y \subset \mathbb{R}^m \text{ open, } g : Y \rightarrow \mathbb{R}^p, \quad g \in C^k$$

Partial Derivatives commute in C^k

$$k \geq 2, \quad X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}^m, \quad f \in C^k$$

$$\forall x, y : \quad \partial_{x,y} f = \partial_{y,x} f$$

This generalizes for $\partial_{x_1, \dots, x_n} f$.

Remark Linearity of Partial Derivatives

$$\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$$

Assuming both $\partial_x f_{1,2}$ exist.

Def Laplace Operator

$$\Delta f := \text{div}(\nabla f(x)) = \sum_{i=0}^n \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \right) = \sum_{i=0}^n \frac{\partial^2 f}{\partial x_i^2}$$

The Hessian

$$X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}, \quad f \in C^2, \quad x_0 \in X$$

$$\mathbf{H}_f(x) := \begin{bmatrix} \partial_{1,1} f(x_0) & \partial_{2,1} f(x_0) & \cdots & \partial_{n,1} f(x_0) \\ \partial_{1,2} f(x_0) & \partial_{2,2} f(x_0) & \cdots & \partial_{n,2} f(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1,n} f(x_0) & \partial_{2,n} f(x_0) & \cdots & \partial_{n,n} f(x_0) \end{bmatrix}$$

$$\text{Where } (\mathbf{H}_f(x))_{i,j} = \partial_{x_i, x_j} f(x)$$

Note that $f : X \rightarrow \mathbb{R}$, i.e. \mathbf{H}_f only exists for 1-dimensionally valued f

$$\text{Notation } \mathbf{H}_f(x) = \text{Hess}_f(x) = \nabla^2 f(x)$$

Remark $\mathbf{H}_f(x_0)$ is symmetric: $(\mathbf{H}_f(x_0))_{i,j} = (\mathbf{H}_f(x_0))_{j,i}$

Def Polar Coordinates

$$g(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

$$\mathbf{J}_g(r, \theta) = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

$$\partial_x f = \cos(\theta) \partial_r f - \frac{1}{r} \sin(\theta) \partial_\theta f$$

$$\partial_y f = \sin(\theta) \partial_r f + \frac{1}{r} \cos(\theta) \partial_\theta f$$

$$(r, \theta) \in (0, +\infty) \times \mathbb{R}, \quad \det(\mathbf{J}_g) = r$$

5.4 Taylor Polynomials

$$\text{Def } |m| := \sum_{i=1}^n m_i$$

$$\text{Def } y^m := y_1^{m_1} \cdots y_n^{m_n}$$

$$\text{Def } m! := m_1! \cdots m_n!$$

$$\text{for } m = (m_1, \dots, m_n), \quad y = (y_1, \dots, y_n)$$

Taylor Polynomials

$$k \geq 1, \quad f : X \rightarrow \mathbb{R}, \quad f \in C^k, \quad x_0 \in X$$

$$T_k f(y; x_0) := \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

Lem. Taylor Approximation

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

Where $f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$
 $k \geq 1, \quad X \subset \mathbb{R}^n \text{ open, } f : X \rightarrow \mathbb{R}, \quad f \in C^k, \quad x_0 \in X$

Remark Taylor polynomials of degree 1, 2:

$$T_1 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} (x_0^\top \cdot \mathbf{H}_f(x_0) \cdot x_0)$$

Method Calculating $T_k f(y; x_0)$ also yields \mathbf{H}_f for $k \geq 2$.

$$T_2 f((x_0, y_0); (x, y)) = \dots + ax^2 + by^2 + cxy$$

$$\implies \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} 2a & c \\ c & 2b \end{bmatrix}$$

Method Taylor Polynomials can be found by combination.

$$\text{Example: } f(x, y) = \underbrace{e^{y^4}}_1 + \underbrace{\sin(xy)}_2 + \underbrace{2xy^2}_3 - \underbrace{\ln(x^2 + 1)}_4, \quad k = 3$$

- $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \implies e^{y^4} \approx 1 + y^4 + \frac{y^8}{2} + \frac{y^{12}}{6}$
Since $k = 3$, discarding all terms with $\deg > 3$ yields: $e^{y^4} \approx 1$
- $\sin(x) \approx x - \frac{x^3}{6} \implies \sin(xy) \approx xy$
- $2xy^2 \approx 2xy^2$ (Since it's already a polynomial, $\deg = 3$)
- $\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \implies \ln(x^2+1) \approx x^2$

$$\text{Thus: } f(x) \approx 1 + xy + 2xy^2 - x^2 = T_3 f((0, 0); (x, y))$$

5.5 Critical Points

Lem. Local Maxima & Minima

$$\left. \begin{array}{l} f(y) \leq f(x_0) \quad \forall y \text{ close} \\ f(y) \geq f(x_0) \quad \forall y \text{ close} \end{array} \right\} \quad \frac{\partial f}{\partial x_i}(x_0) = 0 \quad \forall i \leq n$$

In other words: $df(x_0) = \nabla f(x_0) = 0$

$f : X \rightarrow \mathbb{R}$, $X \subset \mathbb{R}^n$ open, f diff.-able

Def Critical Point

$$x_0 \in X \text{ is critical} \stackrel{\text{def}}{\iff} \nabla f(x_0) = 0$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$ diff.-able

Remark Existence of Maxima/Minima

Don't have to exist if X is open, only if X is compact.

However, for compact sets, the lemma above no longer applies.

Method Critical points on Compact Sets

Decompose $X = X' \cup B$, s.t. X' is open, B is a boundary.

1. Find critical points in X'
2. Check if any $x \in B$ is a maximum/minimum

Def Non-degenerate Critical Point

$$x_0 \in X \text{ non-deg.} \stackrel{\text{def}}{\iff} \det(\mathbf{H}_f(x_0)) \neq 0$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $f \in C^2$, $x_0 \in X$ is critical

Lem. Definiteness of the Hessian

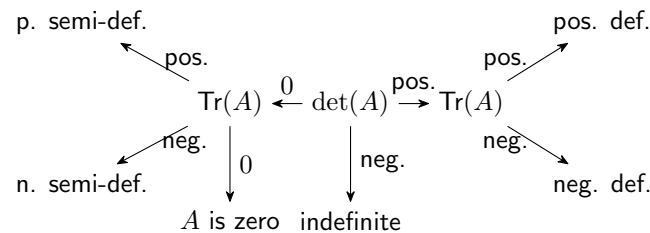
$$\mathbf{H}_f(x_0) \text{ positive definite} \implies x_0 \text{ is a local min.}$$

$$\mathbf{H}_f(x_0) \text{ negative definite} \implies x_0 \text{ is a local max.}$$

$$\mathbf{H}_f(x_0) \text{ indefinite} \implies x_0 \text{ is a saddle point.}$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $f \in C^2$, $x_0 \in X$ non-deg. critical

Method Determining Definiteness for 2×2 Matrices



6 Integral Calculus in \mathbb{R}^n

6.1 Line Integrals

Integrals for $f : I \rightarrow \mathbb{R}^n$

$I = [a, b]$ closed & bounded, $f : I \rightarrow \mathbb{R}^n$ cont.

$$\int_a^b f(t) dt = \left(\int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right)$$

Def Piecewise Continuity

$\exists k \geq 1$, and a Partition $a = t_0 < \dots < t_k = b$

s.t. $f_j : [t_{j-1}, t_j] \rightarrow \mathbb{R}^n$ has $f_j \in C^1$ for all $j \leq k$

For $f : I \rightarrow \mathbb{R}^n$

Def Parametrized Curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ pw.-cont.

Also called *Path* from $\gamma(a)$ to $\gamma(b)$

Line Integral

$\gamma : [a, b] \rightarrow \mathbb{R}^n$ is path, $X \subset \mathbb{R}^n$ s.t. $\gamma([a, b]) \subset X$

$f : X \rightarrow \mathbb{R}^n$ continuous

$$\int_{\gamma} f(s) \cdot ds := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

Def Continuous integrals are linear

$$\int_a^b (f(t) + g(t)) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$$

$f, g : I \rightarrow \mathbb{R}^n$ continuous

Remark $f : X \rightarrow \mathbb{R}^n$ is called a Vector Field.

Def Oriented Reparametrization

For $\gamma : [a, b] \rightarrow \mathbb{R}^n$ (param. curve), $\phi : [c, d] \rightarrow [a, b]$ continuous

$$\sigma : [c, d] \rightarrow \mathbb{R}^n \text{ s.t. } \sigma = \gamma \circ \phi$$

diff.-able on (c, d) , strictly increasing and $\phi(c) = a, \phi(d) = b$

Lem. Oriented Reparametrizations preserve Integrals

$$\int_{\gamma} f(s) \cdot ds = \int_{\sigma} f(s) \cdot ds$$

$\gamma : [a, b] \rightarrow \mathbb{R}^n$ param. curve, σ oriented reparam.,

$\gamma([a, b]) \subset X$, $f : X \rightarrow \mathbb{R}^n$ cont.

Remark Line Integrals of the form $\int_{\gamma} \nabla f(s) \cdot ds$ have:

$$\int_{\gamma} \nabla f(s) \cdot ds = \int_a^b \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\gamma(t)) \gamma'_i(t) = f(\gamma(b)) - f(\gamma(a))$$

Follows from the Chain rule for $h(t) = g(\gamma(t))$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $f \in C^1$, $\gamma : [a, b] \rightarrow X$ param. curve

Def Conservative Vector Field

$f : X \rightarrow \mathbb{R}^n$ conservative $\stackrel{\text{def}}{\iff} \forall \gamma_1, \gamma_2$ s.t. start & end points match:

$$\int_{\gamma_1} f(s) \cdot ds = \int_{\gamma_2} f(s) \cdot ds$$

No matter which path, if start & end match, the integral matches

Remark Closed Curves in Conservative Vector Fields

$$\forall \gamma : [a, a] \rightarrow \mathbb{R}^n : \int_{\gamma} f(s) \cdot ds = 0$$

This is actually equivalent to f being conservative.

The Potential exists in Conservative Vector Fields

$X \subset \mathbb{R}^n$ open, f conservative

$$\exists g \in C^1 : f = \nabla g$$

If $x_1, x_2 \in X$ are joined by a γ , g is unique up to $C \in \mathbb{R}$

$$\nabla g_1 = f \implies g - g_1 \text{ is constant on } X$$

Def Path-Connected Set

$\forall x_1, x_2 \in X : \exists \gamma : [a, b] \rightarrow X$ s.t. $\gamma(a) = x_1, \gamma(b) = x_2$

Lem. Property of Conservative Vector Fields

Easy way to e.g. disprove f being conservative:

$$\forall 1 \leq i \neq j \leq n : \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$, $f \in C^1$, f conserv.

Only this was: This being true does not imply f is conservative!

Def Star Shaped Set

$\exists x_0 \in X : \forall x \in X$ Line seg. $x_0 \rightarrow x$ is in X

Def Convex Set

$\forall x_1, x_2 \in X : \text{Line seg. } x_1 \rightarrow x_2 \text{ is in } X$

Convex implies star shaped.

Th. Some Star Shaped Sets are conservative

In open star-shaped sets $X \subset \mathbb{R}^n$: $f \in C^1$

$$\forall 1 \leq i \neq j \leq n : \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \implies f \text{ conservative}$$

$$\text{Def } \text{curl}(f) := \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix} \quad f : X \rightarrow \mathbb{R}^3, \quad f \in C^1$$

$$\text{Remark } \text{curl}(f) = 0 \iff \forall 1 \leq i \neq j \leq 3 : \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

6.2 The Riemann Integral in \mathbb{R}^n

For $f : X \rightarrow \mathbb{R}$ ($X \subset \mathbb{R}^n$ bounded & closed), $\int_X f(x) dx$ fulfills:

1. Composability

$$\int_X f(x) dx = \int_a^b f(x) dx \quad n=1, X=[a,b]$$

2. Linearity

$$\int_X (af_1(x) + bf_2(x)) dx = a \int_X f_1(x) dx + b \int_X f_2(x) dx$$

f, g cont. on X , $a, b \in \mathbb{R}$

3. Positivity

$$f \leq g \implies \int_X f(x) dx \leq \int_X g(x) dx$$

4. Upper Bound

$$\left| \int_X f(x) dx \right| \leq \int_X |f(x)| dx$$

5. Triangle Inequality

$$\left| \int_X (f(x) + g(x)) dx \right| \leq \int_X |f(x)| dx + \int_X |g(x)| dx$$

6. Volume

$$\int_X f(x) dx \text{ is the volume of } \left\{ (x, y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x) \right\}$$

So the intuitive idea of $\int_a^b f(x) dx$ being the area carries over.

7. Domain Additivity

$$\int_{X_1 \cup X_2} f(x) dx + \int_{X_1 \cap X_2} f(x) dx = \int_{X_1} f(x) dx + \int_{X_2} f(x) dx$$

If X_1, X_2 are compact, f is cont. on $X_1 \cup X_2$

Fubini's Theorem: Multiple Integrals

$$f : X \rightarrow \mathbb{R}, \quad n = n_1 + n_2, \quad n_1, n_2 \geq 1$$

$$X_{x_1} := \left\{ x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X \right\} \subset \mathbb{R}^{n_2}$$

$$X_1 := \left\{ x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset \right\} \subset \mathbb{R}^{n_1}$$

If $g(x_1) := \int_{X_{x_1}} f((x_1, x_2)) dx_2$ is continuous on X_1 :

$$\int_X f(x) dx = \int_{X_1} \left(\int_{X_{x_1}} f((x_1, x_2)) dx_2 \right) dx_1$$

The role of x_1, x_2 can be swapped, if f is continuous.

Def Parametrized m -Set in \mathbb{R}^n

$$f : [a_1, b_1] \times \cdots \times [a_m, b_m] \rightarrow \mathbb{R}^n$$

s.t. $f \in C^1$ on $(a_1, b_1) \times \cdots \times (a_m, b_m)$

A param. 1-set in \mathbb{R}^n is just a param. curve

Def Negligible Subset

$B \subset \mathbb{R}^n$ s.t. $\exists k \geq 0$ param. m_i -sets: $f_i : X_i \rightarrow \mathbb{R}^n$ s.t.

$$B \subset f_1(X_1) \cup \cdots \cup f_k(X_k)$$

$$1 \leq i \leq k, \quad m_i < n$$

Remark For an affine subspace $H \subset \mathbb{R}^n$ with $\dim(H) < n$, any $X \subset \mathbb{R}^n$ contained in H is negligible

Remark The image of a param. curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is negligible.
 γ is a 1-set in \mathbb{R}^n

Lem. Integral of Negligible Sets

For continuous $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}$:

$$X \text{ negligible} \implies \int_X f(x) dx = 0$$

6.3 Improper Integrals

$I \subset \mathbb{R}$ bounded, $J = [a, +\infty]$ for $a \in \mathbb{R}$, f cont. on $X = J \times I$

$$\lim_{x \rightarrow \infty} \int_{[a, x] \times I} f(x, y) dx dy = \underbrace{\int_a^\infty \left(\int_I f(x, y) dy \right) dx}_{\text{Order of Integration may change}}$$

If this Limit is equal for both orders of Integration:

Def Improper Integral in \mathbb{R}^2

$$\int_{J \times I} f(x, y) dx dy := \lim_{x \rightarrow \infty} \int_{[a, x] \times I} f(x, y) dx dy$$

Def Integral over \mathbb{R}^2

$$\int_{\mathbb{R}^2} f(x, y) dx dy := \lim_{R \rightarrow \infty} \int_{[-R, R]^2} f(x, y) dx dy$$

Remark if $|f| \leq g$, and an impr. Integr. exists on g , it exists on f .

6.4 Change of Variable

This is to provide an Analogue of the Change of Variable in \mathbb{R}

$$\int f(g(x))g'(x) dx = \int f(y) dy$$

Prerequisites

$\bar{X}, \bar{Y} \subset \mathbb{R}^n$ compact, $\varphi: \bar{X} \rightarrow \bar{Y}$ cont.

We have: $\bar{X} = X \cup B$, $\bar{Y} = Y \cup C$ s.t.

1. X, Y are open
2. B, C are negligible
3. φ on X is a C^1 map $\varphi: X \rightarrow Y$

Change of Variable in \mathbb{R}^n

\bar{X}, \bar{Y} as above, f cont. on \bar{Y} arbitrary

$$\int_{\bar{X}} f(\varphi(x)) \cdot |\det(\mathbf{J}_{\varphi}(x))| dx = \int_{\bar{Y}} f(y) dy$$

Remark Translations: $\varphi(x) = x + x_0$ have $\mathbf{J}_{\varphi}(x) = \mathbf{I}_n$

so the volume is preserved: $\int_{\bar{X}} f(x + x_0) dx = \int_{x_0 + \bar{X}} f(x) dx$

Remark Linear maps: $\varphi(x) = \mathbf{A}x$ have $\mathbf{J}_{\varphi}(x) = \mathbf{A}$

The change of variable is: $\int_{\bar{X}} f(\varphi(x)) dx = \frac{1}{|\det(\mathbf{A})|} \int_{\bar{Y}} f(y) dy$

Remark Commonly used Changes

1. Polar Coordinates

$$\varphi(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

Where $dx dy = r dr d\theta$

2. Cylindrical Coordinates

$$\varphi(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$$

Where $dx dy dz = r dr d\theta dz$

3. Spherical Coordinates

$$\varphi(r, \theta, \phi) = (r \sin(\phi) \cos(\theta), r \sin(\phi) \sin(\theta), r \cos(\phi))$$

Where $dx dy dz = r^2 \sin(\phi) dr d\theta d\phi$

6.5 Green's Theorem

An analogue of the Fundamental Theorem of Calculus in \mathbb{R}^2 .

Def Simple Closed Parametrized Curve

$\gamma: [a, b] \rightarrow \mathbb{R}^2$ closed param. curve s.t.

1. $\gamma(t) \neq \gamma(s)$ unless $s = t$, or $\{s, t\} = \{a, b\}$
2. $\gamma'(t) \neq 0 \quad \forall a < t < b$

Example: $\varphi(t) = (x_0 + r \cos(t), y_0 + r \sin(t))$

(A circle, traversed *once*, i.e. for $0 \leq t \leq 2\pi$)

Green's Theorem

$X \subset \mathbb{R}^2$ compact with Boundary $\partial X = \bigcup_{1 \leq i \leq n} \gamma_i$ as above

Assume: $\gamma_i: [a_i, b_i] \rightarrow \mathbb{R}^2$ s.t. X is always *left* of $\gamma'_i(t)$ at $\gamma_i(t)$

$$\int_X \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy = \sum_{i=1}^k \int_{\gamma_i} f \cdot ds$$

For a C^1 Vector field $f = (f_1, f_2)$ containing X

So, some Riemann Integrals can be converted to a sum of line integrals.

Lem. Volume using Green

$$\text{Vol}(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot ds = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \cdot \gamma'_{i,2}(t) dt$$

Same assumptions as above.

Remark The *Gauss-Ostrogradski* Formula exists for \mathbb{R}^3 .