

Autonomous Mobile Robots

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1 Introduction

1.1 Probability

Def (Sum rule) $P(X) = \sum P(X, Y) = \sum P(X \cap Y)$

Def (Prod) $P(X, Y) = P(X|Y)P(Y) = P(Y|X)P(X)$

T (Bayes) $P(Y_i|X) = \frac{P(X|Y_i)P(Y_i)}{\sum_{j=1}^n P(X|Y_j)P(Y_j)}$

Def (Cont. Var) Sums become integrals
e.g. $\sum_X P(X) = 1$ becomes $\int p(x) dx = 1$

Def (Indep.) x, y indep. iff $p(x, y) = p(x)p(y)$

Def (Cond. Indep.) iff $p(x, y|z) = p(x|z)p(y|z)$

Def $E[x] = \int_{-\infty}^{\infty} xp(x) dx$, also for $x = f(x)$

Def $\text{Cov}[x] = E[xx^T] - E[x]E[x]^T = \Sigma$

Def (Gauss. Dist.) $x \sim \mathcal{N}(\mu, \Sigma)$ (μ mean, Σ cov.),
PDF: $p(x) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu))$

1.2 Measurement models

$z = b_C + sM_S\omega + b + n + o$: b_C const bias, b time bias, M missal., $n \sim \mathcal{N}(0, R)$ noise, $s\omega$ corr. meas., o other infl.

1.3 Trigonometry

2 Locomotion & Kinematics

2.1 Positioning

Def (Position Vector) ${}^W t_B = {}^W t_W {}^W B$, Original Frame, End point, Target Frame, $\sin = s$, $\cos = c$

Def (State vector) x_R : x, v of rob in W , pos of sensors

Def (Rot. Mat.) $R_z = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 $R_y(\theta) = \begin{bmatrix} c(\psi) & 0 & s(\psi) \\ 0 & 1 & 0 \\ -s(\psi) & 0 & c(\psi) \end{bmatrix} R_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\psi) & -s(\psi) \\ 0 & s(\psi) & c(\psi) \end{bmatrix}$

R Application: ${}_W a = R_{WB} b$

L $R_{BW} = R_{WB}^{-1} = R_{WB}^T$, $\det(R_{WB}) = 1$ (orth.)

R Cols of R_{WB} are basis vec. of Frame \vec{F}_B in \vec{F}_W

Def (Euler Angles) Yaw (z), Pitch (y), Roll (x), mult. rotation matrices, e.g. $R_{EB} = R_z(\psi) \cdot R_y(\theta) \cdot R_x(\varphi)$, **bound.** $[n]^\times = n x^T$ (matrix from vec + arg x)

Def (Rot. Vec) $\alpha = \alpha n$ (n normal)

$R(\alpha, n) = I_3 + \sin(\alpha)[n]^\times + (1 - \cos(\alpha))([n]^\times)^2$

Def (Quaternions) $q = q_w + q_x i + q_y j + q_z k$ with $i^2 = j^2 = k^2 = -1$, $(ij = -ji = k, \text{ same for } jk \text{ and } ki)$

Def (Transf. M) $T_{AB} = \begin{bmatrix} R_{AB} & A t_B \\ 0_{1 \times 3} & 1 \end{bmatrix}$

$T_{BA} = T_{AB}^{-1} = \begin{bmatrix} R_{AB}^T & -R_{AB}^T A t_B \\ 0_{1 \times 3} & 1 \end{bmatrix}$ $T_{AC} = T_{AB} T_{BC}$

2.2 Forward Kinematics (FK)

$T_{WB_n}(\theta) = T_{WB_0} T_{B_0 B_1}(\theta_1) \cdots T_{B_{n-1} B_n}(\theta_n)$.

For 2R system: ${}_W t_{WE} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$

With workspace (pos) W for $\theta_1, \theta_2 \in [-\pi, \pi]$

2.3 Inverse Kinematics (IK)

Option: Solve Forward Kinematics for angles.

Better: Law of cosine with polar coordinates. Compute angle using cosine rule,

$\theta_1 = \phi \pm \alpha$, $\theta_2 = \pm(\pi - \beta)$

(Positive for Elbow Down, Negative for Elbow Up)

Extension to 6R: 1. Waist: spherical coords (2 sol.)

2. 2 sols from 2R for shoulder + elbow

3. Solve for wrist joints (no influence on pos)

2.4 Temporal Models

For **Cont-time n-lin. system of ODE** $\dot{x} = f_C(x(t), u(t))$, with measurements $z(t) = h(x(t)) + v(t)$.

Need linearised (around $f_C(\bar{x}, \bar{u}) = 0$, at **equilibrium**):

$\delta \dot{x}(t) = f_C(\bar{x}, \bar{u}) + F_C \delta x(t) + G_C \delta u(t) + L_C w(t)$

$\delta z(t) = H \delta x(t) + v(t)$. Herein, H is measurements, F_C system, G input gain, w process noise, v measurement noise, both zero-mean **Gaussian White Noise Process**.

For **n-lin. cont-time system**: $\dot{x}(t) = f_C(x(t), u(t), w(t))$

$z(t) = h(x(t)) = v(t)$, linearization is the same

To **discretize**, integrate from t_{k-1} to t_k :

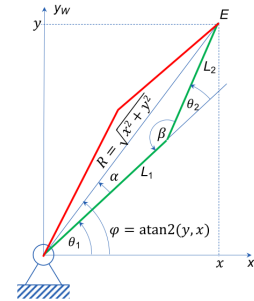
$x_k = f(x_{k-1}, u_k, w_k)$ $z_k = h(x_k) + v_k$, **linearised**:

$\delta x_k = f(\bar{x}, \bar{u}) + F_C \delta x_{k-1} + G_C \delta u_k + L_C w_k$; $\delta z_k = H_C \delta x_k$

Trapezoidal num. int $\Delta x_1 = \Delta t f_C(x_{k-1}, u_{k-1}, t_{k-1})$

$\Delta x_2 = \Delta t f_C(x_{k-1} + \Delta x_1, u_k, t_k)$, then:

$x_k = x_{k-1} + 0.5 \cdot (\Delta x_1 + \Delta x_2)$



2.5 Rigid body & IMU kinematics

Velocity ${}_I v_{IB} = \frac{d}{dt}({}_I t_B)$

Rot. Velocity ${}_I \omega_{IB} = \frac{d}{dt}(\alpha) {}_I t$

Velocity point ${}_B v_{IP} = {}_B v_{IB} +$

${}_B \omega_{IB} \times {}_B t_P$

Rotation Matrices

- For left perturbing $\dot{R}_{IB} = [{}_I \omega_{IB}]^\times R_{IB}$
- For right perturbing $\dot{R}_{IB} = R_{IB} [{}_I \omega_{IB}]^\times$
- Constant angular velocity ($\exp[\Delta \alpha]^\times = \delta R(\Delta \alpha)$)
 $R_{IB}(t + \Delta t) = \exp[\Delta \alpha]^\times R_{IB}(t)$

Quaternions

- For left perturbing $\dot{q}_{IB} = \frac{1}{2} \begin{bmatrix} {}_I \omega_{IB} \\ 0 \end{bmatrix} \otimes q_{IB}$
- For right perturbing $\dot{q}_{IB} = \frac{1}{2} q_{IB} \otimes \begin{bmatrix} {}_B \omega_{IB} \\ 0 \end{bmatrix}$

IMU (Outputs $s\tilde{a}$ (accel.), $s\tilde{\omega}$ (rot. accel.))

${}_W \dot{t}_S = {}_W v$, $\dot{q}_{WS} = \frac{1}{2} q_{WS} \otimes \begin{bmatrix} s\tilde{\omega} + w_g - b_g \\ 0 \end{bmatrix}$

${}_W \dot{v} = R_{WS} (s\tilde{a} + w_a - b_a) + {}_W g$ where gray parts only IRL (in theor. models, leave out), with $\dot{b}_g = w_{b_g}$ and $\dot{b}_a = w_{b_a}$

IMU Sensor Model: $\tilde{z} = b_C + sMz + b + n + o$ where bias b and scale s often modelled time-varying $\dot{b}(t) = \sigma_C n(t)$. b_C const. calib; M Misalignment; n noise; o other infl.

2.6 Rigid Body Dynamics

Def (Newton II) For fin. body w/ mass m and inertia mat. I , with force F and torque T on **Centre of Mass** (CoM), expressed in body frame:

$${}_B F = \sum {}_B F_i = m({}_B \dot{v}_{CoM}) + m{}_B \omega \times {}_B v_{CoM}$$

$${}_B T = \sum {}_B T_i = I({}_B \dot{\omega}) + {}_B \omega \times I{}_B \omega$$

${}_B v_{CoM}$ vel. of CoM, ${}_B \omega$ rot. speed; both w.r.t. world frame

2.7 Wheeled robot Kinematics

Non-holonomic systems not integrable,

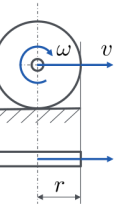
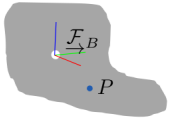
no inst. move in every direct.

Wheel constraints $v_i = \omega_i r_i$

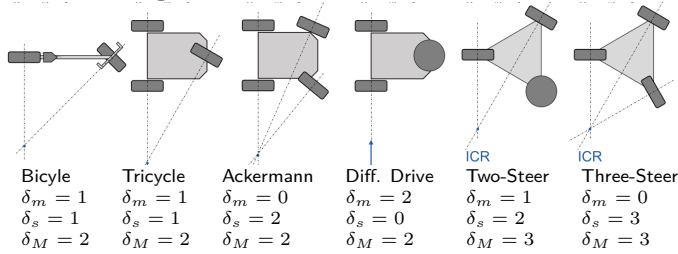
- Driving straight all v equal
- Turning Wheel axis must intersect the **Instant Centre of Rotation** (ICR), speeds: $v_i \div R_i = \Omega$ (R_i dist. wheel-ICR, Ω , vehicle body rotation rate)

Maneuverability

- Deg. of Mobility: $\delta_m = 3 - \# \text{constrained directions}$
- Deg. of Steerability: $\delta_s = \# \text{steerable wheels}$
- Deg. of Maneuverability: $\delta_M = \delta_m + \delta_s$



Wheel Configurations



Differential Drive Kinematics

State vec $x = [x_1, x_2, \theta]^\top$, **Inputs** $u = [\omega_l, \omega_r]^\top$, r_r radius of right wheel, w width of robot

Gen. eq. of Motion $\dot{x}_1 = v \cos(\theta)$, $\dot{x}_2 = v \sin(\theta)$, $\dot{\theta} = \Omega$, with $v = 0.5 \cdot (\omega_l r_l + \omega_r + r_r)$, $\Omega = \frac{\omega_r r_r - \omega_l r_l}{w}$

Straight: $v = \omega_l r_l = \omega_r r_r$, $\Omega = 0$, $D = v \Delta t$.

$$b_s = \begin{bmatrix} D \cos(\theta) \\ D \sin(\theta) \\ 0 \end{bmatrix} \quad b_t = \begin{bmatrix} R(\sin(\Delta\theta + \theta) - \sin(\theta)) \\ -R(\cos(\Delta\theta + \theta) - \cos(\theta)) \\ \Delta\theta \end{bmatrix}$$

Turning: $\Omega = (\omega_l r_l) / R_l = (\omega_r r_r) / R_r$, $R = v / \Omega$, $\Delta\theta = \Omega \Delta t$

Discretized: $x_k = x_{k-1} b_i$ with $i \in \{s, t\}$. ($\int \dots d\Delta t$)

3 Sensors & Actuators

Meas. Model: $z = h(x) + v + o$, with $h(x)$ deterministic mean, v zero-mean noise, o unmodelled effects, x true state

Motor encoders Typ. 64-2048 incrm. per rev; Estim. rot

Rolling-Shutter Most CMOS sensors don't take full image at once, need time stamp for each row

3.1 GNSS

Need ultra-precise time sync ($c \approx 0.3 \text{ m/ns}$). **Errors**

- Multipath problem (signal bounce) (0.5 - 100m)
- Ionosphere delays (10m)
- Satellite pos. err, trop. delay (1m)

3.2 Actuators

Hydraulic acc., easy control, power; maint., speed, price

Pneumatic price, shock abs., speed; acc., loud, maint.

3.2.1 DC Motor

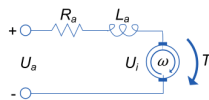
(Kirchoff) $U_a = L_a \dot{I}_a + R_a I_a + U_i$

(Torque, Lorentz Force) $T = k_T I_a$

(Induced V, Faraday) $U_i = k_i \omega$

(Mech. pow. eq. el. pow)

$U_i I_a = k_i \omega I_a = T \omega \Rightarrow k_i = k_T =: k$



3.3 Cameras

Def (Pinhole projection) $\begin{bmatrix} u & v \end{bmatrix}^\top = \frac{f}{z} \begin{bmatrix} x & y \end{bmatrix}^\top$ with f the distance to the lens and z the full distance

$u = c_u + f \cdot x'$ and $v = c_v + f \cdot y'$ where $x' = t_x \div t_z$ and $y' = t_y \div t_z$ where u, v are the pixel x, y coords, $c = [c_u, c_v]^\top$ is optical centre of cam in pixel coords, f scale factor, and ${}^C t_P = [t_x, t_y, t_z]^\top$

The full proj: $u = \begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} f & 0 & c_u \\ 0 & f & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = K {}^C t_P$

If p. in diff frame ${}^w t_P$, then $u = K [R_{CW} {}^C t_{CW}] = {}^w t_P$

3.3.1 Pinhole Camera Projection with distortion

Def Model: $u = k(d(p({}^C t_P)))$, with:

(Projection) $x' = p({}^C t_P) = t_z^{-1} \cdot [t_x, t_y]^\top$

(Distortion model, $r^2 = x'^2 + y'^2$, $x'' = d(x')$)

$$x'' = \frac{1+k_1 r^2+k_2 r^4+k_3 r^6}{1+k_4 r^2+k_5 r^4+k_6 r^6} x' + \frac{2p_1 x' y' + p_2 (r^2 + 2x'^2)}{p_1 (r^2 + 2y'^2) + 2p_2 x' y'}$$

(Scale and Centre) $u = k(x'') = \text{diag}([f_u, f_v]) \cdot x'' + x$

All with k_i radial distortion params, optional for $i > 2$, p_i tang. dist. param, f_u, f_v focal length in pixels

Inverse ${}^C r = [d^{-1}(k^{-1}(u)), 1]^\top$

(To unit plane) $x'' = k^{-1}(u) = [f_u^{-1}, f_v^{-1}]^\top (u - c)$

(Un-distort) $x' = d^{-1}(x'')$ (usually comp. numerically)

(Compute ray) ${}^C r = [x', 1]^\top$

3.3.2 Undestorting a whole image

$u_i = k(d(k_{\text{new}}^{-1}(u_{i,\text{new}})))$ where $u_{i,\text{new}}$ is the place of pixel in output, u_i is the input

Omnidir. Cam undistortion model with $f(u, v) = \sum_{i=0}^N a_i \rho^i$ with $\rho = \sqrt{(u - c_u)^2 + (v - c_v)^2}$, $N = 4$ accurately describes it for most fisheye and catadioptric cameras

3.4 Depth and Range sensing

3.4.1 Triangulation-based

Struct. Light Single cam, single projector: Spatial acc, no worky in bright light, interference with other IR depth cams

Active Stereo 2 cams, 1 proj: worky in bright light, need stereo matching, less accurate, error grows with distance

3.4.2 Classic Stereo

Both images: same plane, focal length, centre, x -axis. Given corresponding pixels $[u_l, v]$ and $[u_r, v]$, $z = \frac{b \cdot f}{u_r - u_l}$ with $u_l = f \cdot \frac{x}{z} + c_u$ and $u_r = f \cdot \frac{x-b}{z} + c_u$

3.4.3 Time of Flight, Projection

No occlusions/shadows, Interference with other dev, multi-path leading to larger distances sensed

Proj. $z = \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} k(d(p({}^C t_P))) \\ [0, 0, 1] {}^C t_P \end{bmatrix}$ **Back:** ${}^C t_P = \begin{bmatrix} dx' \\ d \end{bmatrix}$

3.4.4 Range Sensors

Ultrasonic Typ. freq: 40kHz - 180kHz, Range: 12cm - 5m, Acc: $\approx 2\text{cm}$, rel error $\approx 2\%$ meas. for transp. surf., cheap(ish), Cone wider, reflect. angle dep, wind / currents

LiDAR Time-of-Flight-based, Accuracy, range, works in bright light, Complex, expensive, one timestamp per measurement.

Typical Ranges: up to 100m