

Autonomous Mobile Robots

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1 Introduction

1.1 Probability

Def Sum rule $\mathbb{P}(X) = \sum \mathbb{P}(X, Y) = \sum \mathbb{P}(X \cap Y)$

Def Prod $\mathbb{P}(X \cap Y) = \mathbb{P}(X|Y)\mathbb{P}(Y) = \mathbb{P}(Y|X)\mathbb{P}(X)$

Thm Bayes $\mathbb{P}(Y_i|X) = \frac{\mathbb{P}(X|Y_i)\mathbb{P}(Y_i)}{\sum_{j=1}^n \mathbb{P}(X|Y_j)\mathbb{P}(Y_j)}$

Def Cont. Var Sums become integrals (and vice-versa)

e.g. $\sum_X \mathbb{P}(X) = 1$ becomes $\int \mathbb{P}(x) dx = 1$

Def Indep. x, y indep. iff $\mathbb{P}(X \cap Y) = \mathbb{P}(X)\mathbb{P}(Y)$

Def Cond. Indep. iff $\mathbb{P}(X \cap Y|Z) = \mathbb{P}(X|Z)\mathbb{P}(Y|Z)$

Def $\mathbb{E}[x] = \int_{-\infty}^{\infty} x\mathbb{P}(x) dx$, also for $x = f(x)$

Def Cov $[x] = \mathbb{E}[xx^T] - \mathbb{E}[x]\mathbb{E}[x]^T = \Sigma$

Def Gauss. Dist. $x \sim \mathcal{N}(\mu, \Sigma)$ (μ mean, Σ cov.),

PDF: $f(x) = \frac{1}{\sqrt{(2\pi)^k \det(\Sigma)}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)$

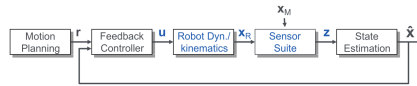
Σ^{-1} for Σ diagonal, inverse of diag els (e.g. σ^{-2} for σ^2 on diags)

Always Normalize (i.e. sum of all probabilities is 1)

1.2 Measurement models

$z = b_C + sM_s\omega + b + n + o$: b_C const bias, b time bias, M missal., $n \sim \mathcal{N}(0, R)$ noise, $s\omega$ corr. meas., o other infl.

Finding: Is in W -frame: may need T_{BW} or R_{BW} . Also see Sec. 3



Feedback loop:

1.3 Trigonometry & Linear Algebra

Def Rule of cosines $c^2 = a^2 + b^2 - 2ab \cos(\gamma)$

Def Orthogonal vec $v^T w = 0$; **Def** Vec Line Eq $p + \lambda d$

Def Determinant $ad - bc$ for mat $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Def Eigenvalues Solve $\det(M - \lambda I) = 0$.

Rem Matrix Inverse For diagonal, inverse of diag els

1.4 Error Propagation

For functions $f(x) = Ax$, the **linear error propagation** is given by $\Sigma^f = A \Sigma^x A^T$, with Σ^x the uncertainty of x (covariance mat.), typically $\text{diag}(\sigma^2)$, with σ^2 the variance of all variables involved

2 Locomotion & Kinematics

2.1 Positioning

Def Pos Vec. ${}^W t_B = {}^W t_{W|B}$, Relative to Frame, P. in other Frame, Start P. of vec in first F., $\sin = s$, $\cos = c$

Def State vector x_R : x, v of rob in W , pos of sensors

Def Rot. Mat. $R_z(\psi)$ (Yaw), $R_y(\theta)$ (Pitch), $R_x(\varphi)$ (Roll)

$$\begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\varphi) & -s(\varphi) \\ 0 & s(\varphi) & c(\varphi) \end{bmatrix}$$

Rem Application: $W a = R_{WBB} a$

Lem $R_{BW} = R_{WB}^{-1} = R_{WB}^T$, $\det(R_{WB}) = 1$ (orth.)

Rem Cols of R_{WB} are basis vec. of Frame $\mathcal{F}_B$ in $\mathcal{F}_W$

Def Euler Ang (Tait-Brian) $R_{EB} = R_z(\psi) \cdot R_y(\theta) \cdot R_x(\varphi)$.

$$\begin{aligned} \psi &= \arcsin(R_{21} / \sqrt{1 - R_{31}^2}) \\ \theta &= \arcsin(-R_{31}) \\ \varphi &= \arcsin(R_{32} / \sqrt{1 - R_{31}^2}) \end{aligned} \quad [n]^\times = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

For pitch axis $\theta = \pm 90$ deg, **Gimbal Lock**, Jacobian **singular**.

Def Angle-Axis $\alpha = \alpha n$ (n normal); To **rotation matrix**:

$$R(\alpha, n) = I_3 + \sin(\alpha)[n]^\times + (1 - \cos(\alpha))([n]^\times)^2$$

To **quaternion**: $q = [n \sin(\alpha/2), \cos(\alpha/2)]$

Def Quaternions $q = q_w + q_x i + q_y j + q_z k$ with $i^2 = j^2 = k^2 = -1$, $(ij = -ji = k, \text{ same for } jk \text{ and } ki)$.

$q = [v(q), a(q)]^T$, $a(q) = q_w$, Add like $\mathbb{C}$ (by "groups")

$$\text{Mult } q \otimes p = \begin{bmatrix} a(q)v(p) + a(p)v(q) + v(q) \times v(p) \\ a(q)a(p) - v(q)^T v(p) \end{bmatrix}$$

To Rot Mat $R_{AB} = I_3 + 2a(q_{AB})[v(q_{AB})]^\times + 2([v(q_{AB})]^\times)^2$

Def Transf. M $T_{AC} = T_{AB} T_{BC}$ Right-Handed typically; Aero: Left-H

$$T_{AB} = \begin{bmatrix} R_{AB} & A^T B \\ 0_{1 \times 3} & 1 \end{bmatrix}; T_{BA} = T_{AB}^{-1} = \begin{bmatrix} R_{AB}^T & -R_{AB}^T A^T B \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

2.2 Forward Kinematics (FK)

$$T_{WB_n}(\theta) = T_{WB_0} T_{B_0 B_1}(\theta_1) \cdots T_{B_{n-1} B_n}(\theta_n).$$

$$\text{For 2R system: } {}^W t_{WE} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Configuration space: $\theta_1, \theta_2 \in [-\pi, \pi]$. **Computation** Similar for nR sys (more angles, more lengths). Last dim is typically sum of angles (or equiv). **For velocity kinematics**: See 2.7.

Def Singularity Loss of deg of Freedom $\det(J(\theta)) = 0$

2.3 Inverse Kinematics (IK)

Option: Solve FK for angles.

Better: Law of cosine with polar coordinates. Compute angle using cosine rule, $\theta_1 = \varphi \pm \alpha$, $\theta_2 = \pm(\pi - \beta)$ (Positive for **Elbow Down**, Neg. for **Elbow Up**)

Extension to 6R: 1. Waist: spherical coords (2 sol.)

2. 2 sols from 2R for shoulder + elbow

3. Solve for wrist joints (no infl. on pos)

For Inv. velocity: $\dot{\theta} = J_S^{-1}(\theta) {}^W \dot{t}_{WE}$

Def Underactuated workspace $< 6D$; IK solvable

Def Overactuated ∞ IK sol; Reduced singularities; Num. sol.

2.4 Temporal Models

Notation Linearization typically written as δX .

Model **robot dyn** as **Cont-time non-lin. system of ODE**:

$$\dot{x} = f_C(x(t), u(t), w(t)), \text{ measurement } z(t) = h(x(t)) + v(t).$$

With: $\frac{\partial}{\partial t} x(t) = f_C(x(t), u(t))$ the model for the robot state update and $h(x(t))$ the model for the measurements (e.g. for IMU)

Linearised at $f_C(\bar{x}, \bar{u}) = 0$, at **equilibrium** (if $\neq 0$, then add it):

$$\delta \dot{x}(t) = F_C \delta x(t) + G_C \delta u(t) + L_C w(t); \delta z(t) = H \delta x(t) + v(t).$$

F_C system (often a Jac. / Taylor, vars are x_i . If no x in F , then linear), G input gain (often Jac / Taylor, vars are u_i , if no u_i , then

lin.), w process noise, H is measurement, v measurement noise, both zero-mean **Gauss. White Noise Proc.** u_k inputs at time k .

Discretized $x_k = f(x_{k-1}, u_k, w_k)$; $z_k = h(x_k) + v_k$

Linearised: $\delta x_k = F \delta x_{k-1} + G \delta u_k + L w_k$; $\delta z_k = H \delta x_k + v(t)$

For discretized, iterative integral from t_{k-1} to t_k

Numerical integration typically happens with one of the below:

Def Euler-Forward $x_k = x_{k-1} + \Delta t f_C(x_{k-1}, u_{k-1}, t_{k-1})$

Def Trapezoidal num. int $\Delta x_1 = \Delta t f_C(x_{k-1}, u_{k-1}, t_{k-1})$

$\Delta x_2 = \Delta t f_C(x_{k-1} + \Delta x_1, u_k, t_k)$, then:

$$x_k = x_{k-1} + 0.5 \cdot (\Delta x_1 + \Delta x_2)$$

2.5 Rigid body & IMU kinematics

Velocity ${}^I v_{IB} = I \dot{t}_B$

Rot. Velocity ${}^I \omega_{IB} = \dot{\alpha} \cdot I n$

Vel. point P ${}^B v_{IP} = {}^B v_{IB} + {}^B \omega_{IB} \times {}^B t_P$

Rotation Matrices

- For left perturbing $\dot{R}_{IB} = [{}^I \omega_{IB}]^\times R_{IB}$
- For right perturbing $\dot{R}_{IB} = R_{IB} [{}^B \omega_{IB}]^\times$
- Constant angular velocity (exp $[\Delta \alpha]^\times = \delta R(\Delta \alpha)$ $\alpha = {}^I \omega_{IB} \delta t$) $R_{IB}(t + \Delta t) = \exp[\Delta \alpha]^\times R_{IB}(t)$

Quaternions

- For left perturbing $\dot{q}_{IB} = \frac{1}{2} \begin{bmatrix} {}^I \omega_{IB} \\ 0 \end{bmatrix} \otimes q_{IB}$
- For right perturbing $\dot{q}_{IB} = \frac{1}{2} q_{IB} \otimes \begin{bmatrix} {}^B \omega_{IB} \\ 0 \end{bmatrix}$

IMU (Outputs $s\tilde{a}$ (accel.), $s\tilde{\omega}$ (rot. vel.))

$${}^w \dot{t}_S = {}^w v; \quad \dot{q}_{WS} = \frac{1}{2} q_{WS} \otimes \begin{bmatrix} s\tilde{\omega} + w_g - b_g \\ 0 \end{bmatrix}$$

${}^w \dot{v} = R_{WS} (s\tilde{a} + w_a - b_a) + w_g$ where gray parts only IRL (in theor. models, leave out), with $\dot{b}_g = w_{b_g}$ and $\dot{b}_a = w_{b_a}$

IMU Sensor Model: $\tilde{z} = b_C + sMz + b + n + o$ where bias b and scale s often modelled time-varying $\dot{b}(t) = \sigma_C n(t)$. b_C const. calibration / bias; M Misalignment; n noise; o other infl.

2.6 Rigid Body Dynamics

Def Newton II For fin. body w/ mass m and inertia mat. I , with force F and torque T on **Centre of Mass** (CoM), in body frame:

$${}^B F = m({}^B \dot{v}_{CoM}) + m({}^B \omega \times {}^B v_{CoM})$$

$${}^B T = I({}^B \dot{\omega}) + {}^B \omega \times I {}^B \omega$$

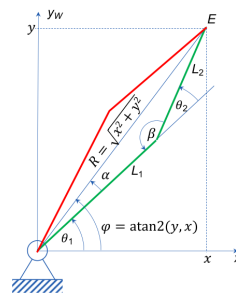
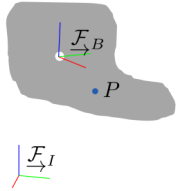
${}^B v_{CoM}$ vel. of CoM, ${}^B \omega$ rot. speed; both w.r.t. inert. frame

To determine dynamics: (1) Define control inputs (e.g. wheel speeds), (2) Assumptions / Simplifications about state & inputs

(3) List forces, torques, etc needed, (4) Express as func of states / inp (5) Reduce to the CoM (6) Plug into Newton-Euler eq combined with rigid body kinematics

2.7 Manipulator Velocity Kinematics

Differentiate FK $x = f(\theta)$, with $\dot{x} = [{}^w v, {}^w \omega]^\top = J(\theta) \dot{\theta}$, with $J = \frac{\partial f}{\partial \theta}$ manipulator Jacobian. Then $\dot{\theta} = J^{-1} \dot{x}$ or $\dot{\theta} = J^+ \dot{x}$, if over-actuated, with $J^+ = J^\top (J J^\top)^{-1}$ (Moore Penrose-Inverse)



2.8 Legged robots

Legged robots don't need to maintain static stability!

Def Static Walking 3 or more legs on ground, stable if frozen, safe, slow, inefficient.

Def Dynamic Walking < 3 legs on ground, falls when not moving, fast, efficient, hard.

2.9 Wheeled robot Kinematics

Non-holonomic systems **not integrable**, no inst. move in every direct. **Wheel constraints** $v_i \div r_i$ (r_i constraints, (wheel radii))

- Driving straight all v equal (ICR: R.Cent.)
- Turning Wheel axis must intersect the **Instant Centre of Rotation** (ICR) of vehicle, speeds: $v_i \div R_i = \Omega$ (R_i = dist. wheel-ICR; Ω : vehicle rotation rate (around ICR))

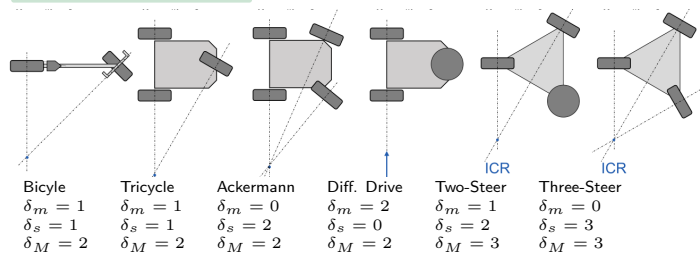
To compute ICR, use $v_i \div R_i = \Omega$ and similarity (see 3.4.2).

Below: α , l pos in frame, β rot at that pos (z -ax). To compute c in $c \cdot B v_{WB} = \omega$ (For multiple wheels, construct mat. from this) $c_i = [\sin(\alpha + \beta) \div r, -\cos(\alpha + \beta) \div r, -\cos(\beta) \cdot l \div r]$

Maneuverability

- Deg. of Mobility: $\delta_m = 3 - \# \text{constrained directions}$
- Deg. of Steerability: $\delta_s = \# \text{steerable wheels}$
- Deg. of Maneuverability: $\delta_M = \delta_m + \delta_s$

Wheel Configurations



2.9.1 Differential Drive Kinematics

State vec $x = [x_1, x_2, \theta]^\top$, **Inputs** $u = [\omega_l, \omega_r]^\top$, wheel radii r_l, r_r , w width of robot

Gen. eq. of Motion $\dot{x}_1 = v \cos(\theta)$, $\dot{x}_2 = v \sin(\theta)$, $\dot{\theta} = \Omega$, with $v = 0.5 \cdot (\omega_l r_l + \omega_r r_r)$, $\Omega = \frac{\omega_r r_r - \omega_l r_l}{w}$

Straight: $v = \omega_l r_l = \omega_r r_r$, $\Omega = 0$, $D = v \Delta t$.

Turning: $v = (\omega_l r_l) / R_l = (\omega_r r_r) / R_r$, $R = v / \Omega$, $\Delta \theta = \Omega \Delta t$

Discretized: $x_k = x_{k-1} + b_i$ with $i \in \{s, t\}$, respectively, with $b_s = \begin{bmatrix} D \cos(\theta) \\ D \sin(\theta) \\ 0 \end{bmatrix}$, $b_t = \begin{bmatrix} R(\sin(\Delta \theta + \theta) - \sin(\theta)) \\ -R(\cos(\Delta \theta + \theta) - \cos(\theta)) \\ \Delta \theta \end{bmatrix}$; R dist ICR

Swedish Wheels: Have 3 DoF, typ. (3 pcs) equally spaced on circle. Typ. Param. each: α, β, γ (z, x, y axes, oriented along x, z up)

Wheel rot speed from movement Typ. comp: $J \cdot R(\theta) \cdot x = \dot{\varphi}$, with $R(\theta) \cdot x$ transform from base frame to robot frame.

3 Sensors & Actuators

Meas. Model: $z = h(x) + v + o$, with $h(x)$ determ. mean, v zero-mean noise, o unmodelled effects, x true state. $h(x)$ describes how to compute x from known values (e.g. Intercept-Theorem). Probabilistic M.M: use random variable in definition, and s. 1.4

Motor encoders Typ. 64-2048 increments per rev; Estimate rot

Rolling-Shutter Most CMOS sensors don't take full image at once, need time stamp for each row

Def Proprioceptive Robot-internal states (e.g. IMU, encoders)

Def Exteroceptive Environment (e.g. LIDAR, Cameras)

3.1 GNSS

Need ultra-precise time sync ($c \approx 0.3 \text{ m/ns}$). **Errors**

- Multipath problem (signal bounce) (0.5 - 100m)
- Ionosphere delays (10m)
- Satellite pos. err, trop. delay (1m)

3.2 Actuators

Hydraulic acc., easy control, power; maint., speed, price

Pneumatic price, shock abs., speed; acc., loud, maint.

3.2.1 DC Motor

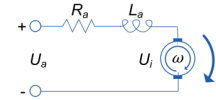
(Kirchoff) $U_a = L_a \dot{I}_a + R_a I_a + U_i$

(Torque, Lorenz Force) $T = k_T I_a$

(Induced V, Faraday) $U_i = k_i \omega$

(Mech. pow. = electric power)

$U_i I_a = k_i \omega I_a = T \omega = k_T I_a \omega \Rightarrow k_i = k_T = k$



3.3 Cameras

Def Pinhole projection $\begin{bmatrix} u & v \end{bmatrix}^\top = \frac{f}{z} \begin{bmatrix} x & y \end{bmatrix}^\top$ with f the distance to the lens and z the distance from object

$u = c_u + f \cdot x' / z$ and $v = c_v + f \cdot y' / z$ where $x' = t_x \div t_z$ and $y' = t_y \div t_z$ where u, v are the pixel x, y coords, $c = [c_u, c_v]^\top$ is optical centre of cam in pixel coords, f scale factor.

The full proj: $u = \begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} f & 0 & c_u \\ 0 & f & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = K c t_P$

If p. in diff frame, e.g. W -frame then $u = K[R_{CW} c t_{CW}]_W t_P$

3.3.1 Pinhole Camera Projection with distortion

Def Model: $u = k(d(p(c t_P)))$, with c as above, k_i radial distortion params, optional for $i > 2$, p_i tangential distortion parameter, f_u, f_v x/y focal lengths in pixels. These should be given by task

(Projection) $x' = p(c t_P) = t_z^{-1} \cdot [t_x, t_y]^\top$

(Distortion model) $r^2 = x'^2 + y'^2$, $x'' = d(x')$

$x'' = \frac{1+k_1 r^2+k_2 r^4+k_3 r^6}{1+k_4 r^2+k_5 r^4+k_6 r^6} \cdot x' + \frac{2p_1 x' y' + p_2(r^2+2x'^2)}{p_1(r^2+2y'^2)+2p_2 x' y'}$

(Scale and Centre) $u = k(x'') = \text{diag}([f_u, f_v]) \cdot x'' + c$

(To unit plane) $x'' = k^{-1}(u) = \text{diag}([f_u^{-1}, f_v^{-1}]) \cdot (u - c)$

(Un-distort) $x' = d^{-1}(x'')$ (usually comp. numerically)

(Compute ray) $c r = [x', 1]^\top$

3.3.2 Undistorting a whole image

$u_i = k(d(k_{\text{new}}^{-1}(u_{i,\text{new}})))$, where $u_{i,\text{new}}$ is the place of pixel in output, u_i is the input, with k_{new} the new focal lengths and image centre

Omnidir. Cam undistortion model with $f(u, v) = \sum_{i=0}^N a_i \rho^i$ with $\rho = \sqrt{(u - c_u)^2 + (v - c_v)^2}$, $N = 4$ accurately describes it for most fisheye and catadioptric cameras

3.4 Depth and Range sensing

Typ. derive D.M. with sensor dir param (as vec line eq)

3.4.1 Triangulation-based

Struct. Light Single cam, single projector: Spatial acc, no worky in bright light, interference with other IR depth cams

Active Stereo 2 cams, 1 proj: worky in bright light, need stereo matching, less accurate, error grows with distance

3.4.2 Classic Stereo

Both images: same plane, focal length f , centre, x -axis. Given corresponding pixels $[u_l, v]$ and $[u_r, v]$: **disparity** (pixel offset) $d = u_r - u_l$; $z = \frac{b \cdot f}{d}$, then projected into left (right) cam, $u_l = f \cdot \frac{x}{z} + c_u$ or $u_r = f \cdot \frac{x-b}{z} + c_u$, b distance between cameras. Then often apply pinhole camera projection (to e.g. get 3D point coords, comp $c r$, then point $z \cdot c r$). Uncertainty introduced with 1.-order error propag. $\Delta z = \frac{\partial}{\partial d} z \Delta d$, with Δd error of d . Too large b disables depth estimations, too small increases error

Thm Intercept $f \div Z = u_l \div X$, with Z and X the z and x coordinates of tracked object

3.4.3 Time of Flight, Projection

No occlusions/shadows, Interference with other dev, multipath leading to larger distances sensed

Proj. $z = \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} k(d(p(c t_P))) \\ [0, 0, 1] c t_P \end{bmatrix}$ **Back:** $c t_P = \begin{bmatrix} d x' \\ d \end{bmatrix}$ This x' is obtained from pinhole cam projection.

3.4.4 Range Sensors

Ultrasonic Typ. freq: 40kHz - 180kHz, Range: 12cm - 5m, Acc: $\approx 2\text{cm}$, rel error $\approx 2\%$ meas. for transp. surf., cheap(ish), Cone wider, reflect. angle dep, wind / currents

LiDAR Time-of-Flight-based, Accuracy, range, works in bright light, Complex, expensive, one timestamp per measurement. Typical Ranges: up to 100m

4 Multi-Sensor Estimation

4.1 Linearization

$f(x) \approx f(\bar{x}) + J_f|_{x=\bar{x}}(x - \bar{x})$, f' , no vec in 1D; $\bar{x}$ lin. p.

Def Jacobian J_f rows for eq of f ; cols for vars of each eq. It may also be a single value (if just one var in the state)

Def Gradient ∇f is vec, each comp. for par diff of var

Part. diff; Approx. using finite differences $\frac{f(\bar{x}+h) - f(\bar{x})}{h}$, or central differences (vector of $\frac{f(\bar{x}+h_i e_i) - f(\bar{x}-h_i e_i)}{h_i}$, with e_i unit vec)

4.2 Linear Least Squares

Goal: $\text{argmin}_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$, A : rows i -th datap. col c : t_i^{c-1} .

Alt: compute sum of squared errors S , then min. S , i.e. solve system $\frac{\partial}{\partial \alpha} S = 0$, $\frac{\partial}{\partial r} S = 0$ for $\{\alpha, r\}$ the params

Man. sol.: comp. $M = A^\top A$, $b' = A^\top b$, then $Mx = b'$.

Numpy: `numpy.linalg.lstsq`

Prob. sol.: $\text{argmax } \mathbb{P}(x|z)$ with Maximum ...

- Likelihood** $\mathbb{P}(x|z) \propto \mathbb{P}(z|x) = \prod_{i=1}^N \mathbb{P}(z_i|x)$

- a Post** $\mathbb{P}(x|z) \propto \mathbb{P}(z|x) \mathbb{P}(x) = \mathbb{P}(x) \prod_{i=1}^N \mathbb{P}(z_i|x)$

Gauss-Dist (non)-linear meas. processed in batches, MLE is weighted non-linear Least Squares

4.3 Non-Linear Least Squares

Find $x^* = \operatorname{argmax} \mathbb{P}(x|z) = \operatorname{argmin}(-\log(\mathbb{P}(x|z)))$

Gauss-Newton Need func F and its Jacobian (or derivative) DF
def gauss_newton(x: np.ndarray, F, DF, tol=1e-6):
 s = np.linalg.lstsq(DF(x), F(x))[0] # least sq.
 x = x-s; k = 1
 while np.linalg.norm(s) > tol * np.linalg.norm(x):
 s = np.linalg.lstsq(DF(x), F(x))[0]
 x = x-s; k += 1 # k optional for max iter
 return x, k

Levenberg-Marquardt (1) Pick start point $\bar{x}^0$ and start param $\lambda^0 = \max \operatorname{diag}(A)$ and v (e.g. $v = 2$).; **(2)** Modified GN sys: $A + \lambda \operatorname{diag}(A) \Delta x = b$; **(3)** Solve for Δx ; **(4)** Update: $\bar{x}^{k+1} = \bar{x}^k + \Delta x$ and $\lambda^{k+1} = \lambda^k \div v$ (if cost reduced), else: $\bar{x}^{k+1} = \bar{x}^k$ and $\lambda^{k+1} = \lambda^k v$, go to step 3; **(5)** Check convergence, else go to step 2

Robust Cost Functions Account for outliers, by mod. err. terms

4.4 Bayes Filter (in DAG)

x_k^R state at time k, z^d dist. meas., u_k^p wheel odometry (= meas.). Typically care about current state: alternate predict & update. Prediction with Product/Sum rule for $\mathbb{P}[x_k^R | u_{1:k}^p, z_{1:k-1}^d]$, Update with Bayes' Theorem for $\mathbb{P}[x_k^R | u_{1:k}^p, z_{1:k}^d]$

4.5 Particle Filter

Is a Bayes filter approximating the state with S weighted particles $\{x_{k,s}, w_{k,s}\}$. Distribution not necessarily unimodal.

Predict: Push each particle through transition model; Update step:

- Apply Bayes rule $w'_{k,s} = \mathbb{P}[z_i | x_{k,s}] w_{k-1,s}$
- Renormalize: $w_{k,s} = w'_{k,s} \div \sum_s w'_{k,s}$
- Resample: rand. sel. S particles acc. to weights, $w_{k,s} = S^{-1}$

4.6 Kalman Filter (KF)

Bayes Filter for Gauss. dist of R.V. & **linear meas. model.**

Initial state $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$, P_0 previous covariance

Prediction With lin. state trans. model (u_k wheel spd, w_k noise, covariance Q_k). $x_k = Fx_{k-1} + Gu_k + Lw_k$ with $w_k \sim \mathcal{N}(0, Q_k)$:

- Mean** $\hat{x}_{k|k-1} = F\hat{x}_{k-1} + Gu_k$
- Covariance** $P_{k|k-1} = FP_{k-1|k-1}F^\top + LQ_kL^\top$

Update Lin. meas.: $\tilde{z}_k = Hx_k + v_k$ with $v_k \sim \mathcal{N}(0, R_k)$:

- Measurement residual**: $y_k = \tilde{z}_k - H\hat{x}_{k|k-1}$
- Residual Covariance**: $S_k = HP_{k|k-1}H^\top + R_k$
- Kalman gain**: $K_k = P_{k|k-1}H^\top S_k^{-1}$
- Updated mean**: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k y_k$
- Updated Cov.**: $P_{k|k} = (I - K_k H)P_{k|k-1}$

All with G the input model, L noise map, $\tilde{z}_k$ actual measurement, K trusts measurement when R_k small (or large, but not in between) relative to $P_{k|k-1}$.

4.7 Extended Kalman Filter (EKF)

Non-l. state trans. model $x_k = f(x_{k-1}, u_k, w_k)$ as above:

- Mean**: $\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k)$
- Covariance**: $P_{k|k-1} = F_k P_{k-1|k-1} F_k^\top + L_k Q_k L_k^\top$

With $F_k = \frac{\partial f}{\partial x}$ and $L_k = \frac{\partial f}{\partial w}$ (computed as are Jacobians)

Update Non-Linear measurement: $\tilde{z}_k = h(x_k) + v_k$:

- Measurement residual**: $y_k = \tilde{z}_k - h(\hat{x}_{k|k-1})$

Difference to KF: H becomes H_k , and $H^\top$ is $H_k^\top$. They are linearizations of h , see 2.4

Limitations Linearization error (shifts bias or sharpens it), fixed by UKF (by passing charact. points around mean through nonlin. func instead of Jac) or Iterative EKF

E Diff-Drive Lawn Mower Known: Radii (r_l and r_r) and track w . Measurements: wheel increments ($\Delta\varphi_l$ and $\Delta\varphi_r$). State $[x, y, \theta]^\top$. Can compute distance, heading change & turn radius.

For **State Transition**: $D = 0.5(r_l \Delta\varphi_l + r_r \Delta\varphi_r)$, rest same as in 2.9.1. F_k is of form $[1, 0, d_1; 0, 1, d_2; 0, 0, 1]$, with $d = b_s$ (with first row negated) or $d = b_t$ for straight and turning, respectively.

Updates are linear (GPS: $[x, y]^\top$ and compass: θ , matrix: I)

5 SLAM to Spatial AI

SLAM = Simultaneous Localization and Mapping

5.1 Keypoints

Corner det. $SSD(\Delta x, \Delta y) \approx [\Delta x \ \Delta y] M [\Delta x \ \Delta y]^\top \lambda_i$ E.V. of M ; κ const 0.04-0.15; $R = \det(M) - \kappa \cdot \operatorname{TR}(M)^2 = \lambda_1 \lambda_2 - \kappa(\lambda_1 + \lambda_2)^2$;

$$M = \sum_{x,y \in P} \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix} = R^\top \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R$$

Blob Detection (I is the image)

Laplacian of Gaussian (LoG): $L = g(x, y, t) * I(x, y)$. Then apply

$$\text{Laplacian Operator } \nabla_{\text{norm}}^2 L = t \left(\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} \right)$$

Diff. of Gaussians (DoG): $\Delta L = L(x, y, t) - L(x, y, kt)$

SIFT Detector (1) Subsample & Blur **(2)** DoG on each res. image

(3) Keypoints extrema in DoG pyramid

BRISK / binary descriptors: Compare pixel intensities at fixed sampling pattern around keypoint. Match by Hamming distance (very fast), **SuperPoint**: CNN learns detector + descriptor

5.2 Bootstrapping

PnP Problem Perspective n -Point Find sol. for camera pose *directly*

RANSAC RANdom SAMpling Consensus Model estimation, for find. outliers & correct. Also great for finding an initial guess for pose. (due to robustness) For N iteration: sample min set, fit model, count inliers (reprojection error < threshold t), keep best, optionally refit on inliers. More outliers $\Rightarrow$ more iterations.

Stereo Triang. Given two rays (known poses for points in 2D). Find good point in 3D. Fast sol: **Midpoint Method**:

1 Find p. along ray w/ min. dist (Lin. Least Squares)

$$\lambda = [\lambda_1 \ \lambda_2]^\top = \operatorname{argmin} ||(w^t c_2 + \lambda_2 w e_2) - (w^t c_1 + \lambda_1 w e_1)||^2$$

2 Solve normal equation $A\lambda = b$ with $q = -w e_1^\top w e_2$:

$$A = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix} \quad b = \begin{bmatrix} e_1^\top \cdot (w^t c_2 - w^t c_1) \\ -e_2^\top \cdot (w^t c_2 - w^t c_1) \end{bmatrix} \quad C_i \text{ cam}$$

3 Pick midp. $w^t p = 0.5(\tau_1 + \tau_2)$; $\tau_n = w^t c_n + \lambda_n w e_n$

5.3 Place Recognition

Idea: Build vocab from "visual words" (in Training). **Runtime** Detect keypoints; Extract descriptors; Build histogram; Query DB for similarity; If no match, insert into DB, else use to compute with e.g. RANSAC

5.4 Mapping

Problem Formulations Localisation (always static given map):

$x_{R,k}^* = \operatorname{argmax} \mathbb{P}(x_{R,k} | x_M, z_{1:k}, u_{1:k})$ (recursive)

$x_{R,1:k}^* = \operatorname{argmax} \mathbb{P}(x_{R,1:k} | x_M, z_{1:k}, u_{1:k})$ (batch)

SLAM $\{x_{R,k}^*, x_M^*\} = \operatorname{argmax} \mathbb{P}(x_R, x_M | z_{1:k}, u_{1:k})$ (as $\uparrow$)

Mapp.: $x_M^* = \operatorname{argmax} \mathbb{P}(x_M | x_{R,1:k}^*, z_{1:k}, u_{1:k})$ with given poses $x_{R,1:k}^*$. Thus temp. model $u_{1:k}$ doesn't matter.

Prob. Occ. Grid $\mathbb{P}(f_j) = \mathbb{P}(\neg o_j) = 1 - \mathbb{P}(o_j)$, with $\mathbb{P}(o_j)$ prob. cell j occupied; pairwise independent.

Occ. Map. w/ depth sensor $\mathbb{P}(o_j | x_{R,1:k}) =$

$$\frac{\mathbb{P}(o_j | x_{R,k}, z_k) \mathbb{P}(z_k | x_{R,k}) \mathbb{P}(o_j | x_{R,1:k-1}, z_{1:k-1})}{\mathbb{P}(o_j) \mathbb{P}(z_k | x_{R,1:k}, z_{1:k-1})}$$

map prior; Prev. occ. est.; Occ. based on curr. range meas.;

Update function: $l(a) := \log(\operatorname{Odds}(a))$ with $\operatorname{Odds}(a) = \frac{\mathbb{P}(a)}{1 - \mathbb{P}(a)}$

(inv. sensor model): $l(o_j | x_{R,1:k}, z_{1:k}) =$

$$l(o_j | x_{R,k}, z_k) + l(o_j | x_{R,1:k-1}, z_{1:k-1}) - l(o_j)$$

In 3D 3D voxel j as signed dist. s and weight w , update: $s_k = \frac{w_{k-1} s_{k-1} + \tilde{s}_k}{w_{k-1} + 1}$ with $w_k = \min(w_{\max}, w_{k-1} + 1)$

Implementation Using Hash maps or octree (dense grid inefficient)

5.4.1 Iterative Closest Point (ICP)

Build *correspondences*: associate all live scan points l_i to closest map points m_i . Error term: $e = T_{WL_i L_i} m_i - w l_i$. Minimize this via Gauss-Newton, then re-associate, iterate.

Photometric: $u_L = \pi(T_{WCKF}^{-1} T_{WCL} \pi^{-1}(u_{KF}, D_{KF}[u_{KF}]))$, error term $e = I_{KF}[u_{KF}] - I_L[u_L]$, where all subscript L are from live image, all subscript KF key frame.

5.5 Dense Tracking and Loop Closure

Idea: Min. sum of sq. errors over all pixels w/ Gauss-Newton.

6 Planning & Control

6.1 SISO & MIMO

Single-Input-Single-Output: r Reference (e.g. speed), u Sys. Input (e.g. gas pedal pos), z Sys. Out. (e.g. speed of car)

Multiple-Input-Multiple-Output: r Reference (typ: trajectory), u Sys. Input (e.g. 4 rotor speeds), x internal states (pos, orient, speed, rot. speed), z Sys. Output (e.g. pose, speed)

6.2 Proportional-Integral-Differential (PID)

$u(t) = K_p e(t) + K_i \int_{t_0}^t e(\tau) d\tau + K_d \frac{de(t)}{dt}$, where params K_p (curr), K_i (long-term), K_d (trend) reduce corresp. errors, with $e = r - y$
Drone Control ${}_B F$ and ${}_B M$ as in sect. 2.6, use $u' = {}_B M$.

6.3 Linear Quadratic Regulator (LQR)

For lin. cont.-time dyn. $\dot{x}(t) = F_c x(t) + G_c u(t)$.

We try to minimise cost functional (for $u(t) = -Kx(t)$)

$$J = \int_t^\infty x(\tau)^\top Q x(\tau) + u(\tau)^\top R u(\tau) d\tau$$

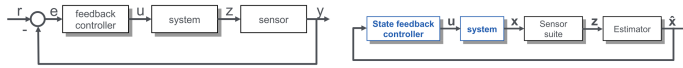
Solution: $K = R^{-1}(G_c^\top P)$, with P found from

$$F_c^\top P + P F_c - (P G_c) R^{-1} (G_c^\top P) + Q = 0$$

Finding K is expensive, but *offline*, at runtime only $u(t)$.

Non-Lin: Approx, $\delta \dot{x}(t) = F_c \delta x(t) + G_c \delta u(t)$

Below feedback loops for PID and LQR



6.4 MPC

Cost function ($p(x_N)$ terminal cost, sum the stage cost)

$$J_{0 \rightarrow N}(x_0, u_0, \dots, u_{N-1}) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

We minimize the above s.t. for $k \in \{0, \dots, N-1\}$ we have $x_{k+1} = f(x_k, u_k)$, $g(x_k, u_k) \leq 0$ and $x_N \in \mathcal{X}_f$ and $x_0 = x(0)$

Finite-Horizon Lin-Quad Control Quad. Cost:

$$J_{0 \rightarrow N}(x_0, u_0, \dots) = x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k$$

without constraints, **State Feedback Law** $u_0^* = -Kx(0)$. With constraints, minimize as above, $x_{k+1} = Fx_k + Gu_k$

6.5 Motion Planning

Config. Space: $\mathcal{C}$. Subset of robot states for planning.

Free C-Space: $\mathcal{C}_{\text{free}}$, **C-Space Obstacles** $\mathcal{C}_{\text{obst}}$ (occupied)

Collision checker: $c(x) : \mathcal{C} \rightarrow \{0, 1\}$

Visibility graph: Connect corners, goal outside obstacles

Voronoi Diagram: Edges at max. dist. from obst. (benefit: safer paths).

Discrete via graph/grid: complete solution, Curse of dimensions

Continuous probabilistic: High-D okay, differential constraints ok, Probabilistic, depending on setup may not find correct sol. or run forever

6.5.1 A* Algorithm

Function $h(\dots)$ is a lower bound of optimal cost. Tends to be faster than Dijkstra.

```

1 procedure ASTAR(Graph, start, goal)
2   for each v in Graph.Vertices do
3     dist[v] ← ∞
4     totDistEst[v] ← ∞
5     prev[v] ← undef
6   insert start into openSet
7   dist[start] ← 0
8   totDistEst[start] ← h(start, goal)
9   while openSet not empty do
10    u ← vert in openSet w/ min totDistEst
11    remove u from openSet
12    if u == goal then
13      return dist[u], prev
14    for each neighbour v of u do
15      alt ← dist[u] + G.Edges(u, v).dist
16      if alt < dist[v] then
17        dist[v] ← alt
18        totDistEst[v] ← alt + h(v, goal)
19        insert v into openSet
20        prev[v] ← u
21   return ∞, prev

```

6.5.2 Exploration

Find action sequence by finding goals, planning collision-free paths there, choose goal/path with maximum utility (typically max map information)

6.5.3 Rapidly-Exploring Random Tree (RRT)

```

1 procedure RRT(start, goal)
2   INSERTVERTEX(start, Graph)
3   for k < MAX_ITER do
4     s ← scal. unif. rand. sample on [0, 1]
5     if s < GOAL_CONNECT_PROB then
6       x ← goal
7     else
8       x ← random coll.-free config
9       x_n ← NEARESTCONFIGURATION(Graph, x)
10      x_f ← STOPPINGCONFIGURATION(x_n, x)
11      if x_f ≠ x_n then
12        insertVertexGraph, x_f
13        insertEdgeGraph, x_n, x_f
14      if x_f == goal then
15        return SUCCESS, Graph
16      return FAILURE, Graph

```

▷ Extension of RRT* goes here

Notes: nearestConfiguration may be on edge, then edge is split; stoppingConfiguration returns furthest config x_f on segment x_n to x that produces collision-free edge (x_n, x_f)

Returns a collision-free path as graph. Need nearest neighbour search. Extension to RRT* to make path better:

```

1 X_near ← NEIGHBOURS(Graph, x_f, R)
2 x_min ← x_n
3 c_min ← COST(x_n) + EDGE_COST(x_n, x_f)
4 for each x_near in X_near do
5   if EDGE_COLLISIONFREE(x_near, x_f) and
6     COST(x_near) + EDGE_COST(x_near, x_f) < c_min then
7     x_min ← x_near
8     c_min ← COST(x_near) + EDGE_COST(x_near, x_f)
9 INSERTEDGE(Graph, x_min, x_f)
10 COST(x_f) ← c_min
11 for each x_near in X_near do
12   if EDGE_COLLISIONFREE(x_f, x_near) and
13     COST(x_f) + EDGE_COST(x_f, x_near) < COST(x_near) then
14     INSERTEDGE(Graph, x_f, x_near)
15     x_parent ← PARENT(Graph, x_near)
16     REMOVEEDGE(Graph, x_parent, x_near)

```

▷ new connection

Informed RRT* extension: Once conn. betw. start and goal found, restrict sampling to (hyper)ellipsoid. $b = 0.5 \sqrt{d^2 - \|x_s - x_g\|^2}$.

6.5.4 Collision Avoidance

Dynamic Window Approach Assume: Robot moves inst. on circ. arcs (v, ω) . Compute arcs with coll. Accounts for Kino-Dyn, Cost func prone to loc. min, assumes static obj

Velocity Obstacles Assume: Robot in str. line (v_x, v_y) . Comp pos with coll. Vel. of obj, prone to local optima, no kino-dyn.

Potential Field Methods Define repulsive and attractive potential $c = c_{\text{att}} + c_{\text{rep}}$. With e.g. $c_{\text{att}} = \frac{1}{2} k_{\text{att}} \|x - x_{\text{goal}}\|^2$ and $c_{\text{rep}} = \begin{cases} \frac{1}{2} k_{\text{rep}} \left(\frac{1}{\rho(x)} + \frac{1}{\rho_{\text{lim}}} \right)^2 & \rho \leq \rho_{\text{lim}} \\ 0 & \text{else} \end{cases}$ Simple control laws, may trap in

loc. min., no diff. const, no guar. to avoid coll

6.6 Learning To Act

Used if there is no state-transition model or cost/reward func.

6.6.1 Markov Decision Process

The goal is to maximize the reward. Along the route, get small reward, at the end large reward (good or bad).

Def. by states $x \in \mathcal{X}$ (RL: s), actions $u \in \mathcal{U}$ (RL: a), prob. state trans. $\mathcal{T}(x, u, x_+) = \mathbb{P}(x_+ | x, u)$, reward func $\mathcal{R}(x, u, x_+)$, start state x_0 , optional terminal state x_N .

Utility Func Expected reward: $V = \sum_{k=0}^{\infty} r_k$ or discounted reward $V = \sum_{k=0}^{\infty} \gamma^k r_k$ with $\gamma < 1$ (if inf. runtime)

Solving (Val iter) $V_0(x) = 0$ and $V_{i+1}(x) = \max_u Q(x, u)$ with

$$Q(x, u) = \sum_{x_+} \mathbb{P}(x_+ | x, u) [\mathcal{R}(x, u, x_+) + \gamma V_i(x_+)]$$

Repeat until conv. to V^* ($\mathcal{O}(|\mathcal{U}||\mathcal{X}|^2)$ per iter). Optimal policy:

$$\pi^*(x) = \operatorname{argmax}_u Q(x, u)$$

Using **policy iter**:

```

1 Choose  $\pi_0(x)$ 
2 while policy has not converged do
3   repeat  $V_{i+1}^{\pi_j}(x) = Q(x, \pi(x)) \forall x$  and fixed pol.  $\pi_j$ 
4   until values converge
5 One step:  $\pi_{j+1}(x) = \operatorname{argmax}_u Q(x, u)$  with  $V_i = V_{i+1}^{\pi_j}$ 

```

Model-based learning uses empirical models of $\mathcal{T}$ and $\mathcal{R}$

6.6.2 Reinforcement Learning

Passive Direct Evaluation Act according to policy π , store sum of discounted rewards, average them. (But too simple)

Sample-Based Use $V_{i+1}^{\pi}(x) = Q(x, \pi(x))$ w/ $V_0^{\pi}(x) = 0$. We need state transition model for that (we don't have that), instead $\tilde{R}_j$ (approx. probability w/ statistics) and thus

$$V_{i+1}^{\pi}(x) = \frac{1}{N} \sum_{j=1}^N \left(\tilde{R}_j(x, \pi(x), x_+) + \gamma V_i^{\pi}(x_+) \right)$$

Active Find optimal policy π instead of state values $V(x)$.

Q-Learn. Here: restate Val. Iter in Q-Values ($Q_0(x, u) = 0$):

$$Q_{i+1}(x, u) = Q(x, u) + \frac{1}{N} \sum_{x_+} \left(\tilde{R}_j(x, u, x_+) + \gamma \max_u Q_i(x_+, u) - Q_i(x, u) \right)$$

Compute using sample:

$$\tilde{Q}_i(x, u) = \tilde{R}_i(x, u, x_+) + \gamma \max_u Q_i(x_+, u)$$

Then update: $Q_{i+1}(x, u) = (1 - \alpha) Q_i(x, u) + \alpha \tilde{Q}_i(x, u)$. Called off-policy learning, needs exploration. Simplest is random actions (ε -greedy): ε is prob. to act randomly, $1 - \varepsilon$ is prob. to act on pol. Space explored, still doing random stuff

DL-Approaches Model-based (estimate trans. model, e.g. Dyna), Value-based (estimate val or Q-func and extract pol., e.g. Q-Learn), Actor-Critic (estim. val or Q of curr. pol., improve pol., e.g. A3C, SAC), Policy-Gradient (diff. expect. reward w.r.t. params of policy network, e.g. REINFORCE)

