

Autonomous Mobile Robots

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1 Introduction

1.1 Probability

Def Sum rule $\mathbb{P}(X) = \sum \mathbb{P}(X, Y) = \sum \mathbb{P}(X \cap Y)$

Def Prod $\mathbb{P}(X \cap Y) = \mathbb{P}(X|Y)\mathbb{P}(Y) = \mathbb{P}(Y|X)\mathbb{P}(X)$

Thm Bayes $\mathbb{P}(Y_i|X) = \frac{\mathbb{P}(X|Y_i)\mathbb{P}(Y_i)}{\sum_{j=1}^n \mathbb{P}(X|Y_j)\mathbb{P}(Y_j)}$

Def Cont. Var Sums become integrals

e.g. $\sum_X \mathbb{P}(X) = 1$ becomes $\int \mathbb{P}(x) dx = 1$

Def Indep. x, y indep. iff $\mathbb{P}(X \cap Y) = \mathbb{P}(X)\mathbb{P}(Y)$

Def Cond. Indep. iff $\mathbb{P}(X \cap Y|Z) = \mathbb{P}(X|Z)\mathbb{P}(Y|Z)$

Def $\mathbb{E}[x] = \int_{-\infty}^{\infty} x\mathbb{P}(x) dx$, also for $x = f(x)$

Def Cov $[x] = \mathbb{E}[xx^T] - \mathbb{E}[x]\mathbb{E}[x]^T = \Sigma$

Def Gauss. Dist. $x \sim \mathcal{N}(\mu, \Sigma)$ (μ mean, Σ cov.),
PDF: $f(x) = \frac{1}{\sqrt{(2\pi)^k \det(\Sigma)}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)$

Σ^{-1} for Σ diagonal, inverse of diag els (e.g. σ^{-1})

1.2 Measurement models

$z = b_C + sM_s\omega + b + n + o$: b_C const bias, b time bias, M missal., $n \sim \mathcal{N}(0, R)$ noise, $s\omega$ corr. meas., o other infl.

Finding: Is in W -frame: may need T_{BW} or R_{BW} . Also see Sec. 3

1.3 Trigonometry & Linear Algebra

Def Rule of cosines $c^2 = a^2 + b^2 - 2ab \cos(\gamma)$

Def Orthogonal vec $v^T w = 0$; **Def** Vec Line Eq $p + \lambda d$

Def Determinant $ad - bc$ for mat $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1.4 Error Propagation

For functions $f(x) = Ax$, the **linear error propagation** is given by $\Sigma^f = A\Sigma^x A^T$, with Σ^x the uncertainty of x (covariance mat.)

2 Locomotion & Kinematics

2.1 Positioning

Def Pos Vec. $\boxed{W} t \boxed{B} = \boxed{W} t \boxed{W} \boxed{B}$, Relative to Frame, P. in other Frame, Start P. of vec in first F., $\sin = s$, $\cos = c$

Def State vector x_R : x, v of rob in W , pos of sensors

Def Rot. Mat. $R_z(\psi)$ (Yaw), $R_y(\theta)$ (Pitch), $R_x(\varphi)$ (Roll)

$$\begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\varphi) & -s(\varphi) \\ 0 & s(\varphi) & c(\varphi) \end{bmatrix}$$

Rem Application: $w_a = R_{WBB} a$

Lem $R_{BW} = R_{WB}^{-1} = R_{WB}^T$, $\det(R_{WB}) = 1$ (orth.)

Rem Cols of R_{WB} are basis vec. of Frame $\mathcal{F}_B$ in $\mathcal{F}_W$

Def Euler Ang (Tait-Brian) $R_{EB} = R_z(\psi) \cdot R_y(\theta) \cdot R_x(\varphi)$.

$$\begin{aligned} \psi &= \arcsin(R_{21} \div \sqrt{1 - R_{31}^2}) \\ \theta &= \arcsin(-R_{31}) \\ \varphi &= \arcsin(R_{31} \div \sqrt{1 - R_{31}^2}) \end{aligned} \quad [n]^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

For pitch axis = ± 90 deg, Gimbal Lock, Jacobian singular.

Def Angle-Axis $\alpha = \alpha n$ (n normal); Convert to rot. mat

$$R(\alpha, n) = I_3 + \sin(\alpha)[n]^\times + (1 - \cos(\alpha))([n]^\times)^2$$

To quat: $q = [n, \alpha]$

Def Quaternions $q = q_w + q_x i + q_y j + q_z k$ with $i^2 = j^2 = k^2 = -1$, $(ij = -ji = k, \text{ same for } jk \text{ and } ki)$.

$q = [v(q), a(q)]^T$, $a(q) = q_w$, Add like $\mathbb{C}$ (by "groups")

$$\text{Mult } q \otimes p = \begin{bmatrix} a(q)v(p) + a(p) + v(q) \times v(p) \\ a(q)a(p) - v(q)^T v(p) \end{bmatrix}$$

To Rot Mat $R_{AB} = I_3 + 2a(q_{AB})[v(q_{AB})]^\times + 2([v(q_{AB})]^\times)^2$

Def Transf. M $T_{AC} = T_{AB}T_{BC}$ R-Handed typ; Aero: Left-H

$$T_{AB} = \begin{bmatrix} R_{AB} & A^t B \\ 0_{1 \times 3} & 1 \end{bmatrix}; T_{BA} = T_{AB}^{-1} = \begin{bmatrix} R_{AB}^T & -R_{AB}^T A^t B \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

2.2 Forward Kinematics (FK)

$$T_{WBn}(\theta) = T_{WB0}T_{B0B1}(\theta_1) \cdots T_{B_{n-1}B_n}(\theta_n).$$

$$\text{For 2R system: } {}^w t_{WE} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Workspace W : $\theta_1, \theta_2 \in [-\pi, \pi]$. Similar for nR sys (more angles, more lengths). For 2D move in 3D space, last dim is sum of angles (or equiv). The Jacobian of this nD position is typically computed (nD pos is a vec). **Jacobian**: see 4.1.

Def Singularity Loss of deg of Freedom $\det(J(\theta)) = 0$

2.3 Inverse Kinematics (IK)

Option: Solve FK for angles.

Better: Law of cosine with polar coordinates. Compute angle using cosine rule, $\theta_1 = \varphi \pm \alpha$, $\theta_2 = \pm(\pi - \beta)$ (Positive for Elbow Down, Neg. for Elbow Up)

Extension to 6R: 1. Waist: spherical coords (2 sol.)

2. 2 sols from 2R for shoulder + elbow

3. Solve for wrist joints (no infl. on pos)

2.4 Temporal Models

Model **robot dyn** as **Cont-time n-lin. system of ODE** $\dot{x} = f_C(x(t), u(t), w(t))$, with meas. $z(t) = h(x(t)) + v(t)$.

Linearize around $f_C(\bar{x}, \bar{u}) = 0$, at **equilibrium**:

$$\delta \dot{x}(t) = f_C(\bar{x}, \bar{u}) + F_C \delta x(t) + G_C \delta u(t) + L_C w(t)$$

$\delta z(t) = H \delta x(t) + v(t)$. H is meas., F_C system, G input gain, w process noise, v measurement noise, both zero-mean **Gaussian White Noise Process**.

To **discretize**, integrate from t_{k-1} to t_k :

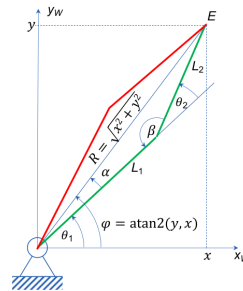
$x_k = f(x_{k-1}, u_k, w_k)$; $z_k = h(x_k) + v_k$, **linearised**:

$$\delta x_k = f(\bar{x}, \bar{u}) + F \delta x_{k-1} + G \delta u_k + L w_k; \delta z_k = H \delta x_k$$

Def Trapezoidal num. int $\Delta x_1 = \Delta t f_C(x_{k-1}, u_{k-1}, t_{k-1})$

$\Delta x_2 = \Delta t f_C(x_{k-1} + \Delta x_1, u_k, t_k)$, then:

$$x_k = x_{k-1} + 0.5 \cdot (\Delta x_1 + \Delta x_2)$$



2.5 Rigid body & IMU kinematics

Velocity ${}^I v_{IB} = {}^I \dot{t}_B$

Rot. Velocity ${}^I \omega_{IB} = \dot{\alpha} \cdot {}^I n$

Velocity point P ${}^B v_{IP} = {}^B v_{IB} + {}^B \omega_{IB} \times {}^B t_P$

Rotation Matrices

- For left perturbing $\dot{R}_{IB} = [{}^I \omega_{IB}]^\times R_{IB}$
- For right perturbing $\dot{R}_{IB} = R_{IB} [{}^I \omega_{IB}]^\times$
- Constant angular velocity ($\exp[\Delta \alpha]^\times = \delta R(\Delta \alpha)$)
 $R_{IB}(t + \Delta t) = \exp[\Delta \alpha]^\times R_{IB}(t)$ $\alpha = {}^I \omega_{IB} \Delta t$

Quaternions

- For left perturbing $\dot{q}_{IB} = \frac{1}{2} \begin{bmatrix} {}^I \omega_{IB} \\ 0 \end{bmatrix} \otimes q_{IB}$
- For right perturbing $\dot{q}_{IB} = \frac{1}{2} q_{IB} \otimes \begin{bmatrix} {}^B \omega_{IB} \\ 0 \end{bmatrix}$

IMU (Outputs $s\tilde{a}$ (accel.), $s\tilde{\omega}$ (rot. accel.))

$${}^w t_S = {}^w v; \quad \dot{q}_{WS} = \frac{1}{2} q_{WS} \otimes \begin{bmatrix} s\tilde{\omega} + w_g \\ 0 \end{bmatrix}$$

${}^w \dot{v} = R_{WS} (s\tilde{a} + w_a - b_a) + {}^w g$ where gray parts only IRL (in theor. models, leave out), with $b_g = w_{b_g}$ and $b_a = w_{b_a}$

IMU Sensor Model: $\tilde{z} = b_C + sMz + b + n + o$ where bias b and scale s often modelled time-varying $\dot{b}(t) = \sigma_C n(t)$. b_C const. calib; M Misalignment; n noise; o other infl.

2.6 Rigid Body Dynamics

Def Newton II For fin. body w/ mass m and inertia mat. I , with force F and torque T on **Centre of Mass** (CoM), in body frame:

$${}^B F = \sum {}^B F_i = m({}^B \dot{v}_{CoM}) + m {}^B \omega \times {}^B v_{CoM}$$

$${}^B T = \sum {}^B T_i = I({}^B \dot{\omega}) + {}^B \omega \times I {}^B \omega$$

${}^B v_{CoM}$ vel. of CoM, ${}^B \omega$ rot. speed; both w.r.t. inert. frame

Legged robots don't need to maintain stability!

To determine dynamics: (1) Define control inputs (e.g. wheel speeds), (2) Assumptions / Simplifications about state & inputs (3) List forces, torques, etc needed, (4) Express as func of states / inp (5) Reduce to the CoM (6) Plug into Newton-Euler eq combined with rigid body kinematics

2.7 Wheeled robot Kinematics

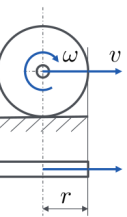
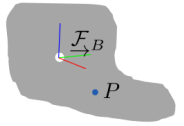
Non-holonomic systems not integrable, no inst. move in every direct.

Wheel constraints $v_i = \omega_i r_i$ (r_i constraints)

- Driving straight all v equal (ICR: R.Cent.)
- Turning Wheel axis must intersect the **Instant Centre of Rotation** (ICR) of vehicle, speeds: $v_i \div R_i = \Omega$ (R_i = dist. wheel-ICR; Ω : vehicle rotation rate (around ICR))

To compute ICR, use $v_i \div R_i = \Omega$ and similarity.

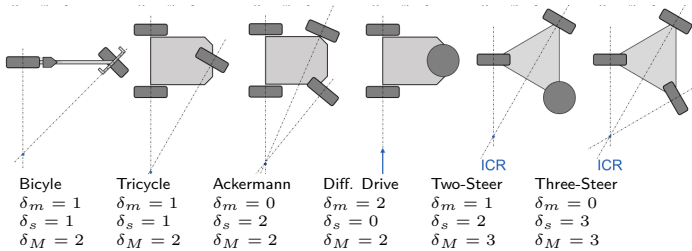
Below: α, l pos in frame, β rot at that pos (z -ax). To compute c in $c \cdot {}^B v_{WB} = \omega$ (For multiple wheels, construct mat. from this)
 $c_i = [\sin(\alpha + \beta) \div r, -\cos(\alpha + \beta) \div r, -\cos(\beta) \cdot l \div r]$



Maneuverability

- Deg. of Mobility: $\delta_m = 3 - \# \text{constrained directions}$
- Deg. of Steerability: $\delta_s = \# \text{steerable wheels}$
- Deg. of Maneuverability: $\delta_M = \delta_m + \delta_s$

Wheel Configurations



Def Differential Drive Kinematics

State vec $\mathbf{x} = [x_1, x_2, \theta]^\top$, **Inputs** $\mathbf{u} = [\omega_l, \omega_r]^\top$, radius of right (left) wheel r_r (r_l), w width of robot

Gen. eq. of Motion $\dot{x}_1 = v \cos(\theta)$, $\dot{x}_2 = v \sin(\theta)$, $\dot{\theta} = \Omega$, with $v = 0.5 \cdot (\omega_l r_l + \omega_r r_r)$, $\Omega = \frac{\omega_r r_r - \omega_l r_l}{w}$

Straight: $v = \omega_l r_l = \omega_r r_r$, $\Omega = 0$, $\dot{D} = v \Delta t$.

$$\mathbf{b}_s = \begin{bmatrix} D \cos(\theta) \\ D \sin(\theta) \\ 0 \end{bmatrix} \quad \mathbf{b}_t = \begin{bmatrix} R(\sin(\Delta\theta + \theta) - \sin(\theta)) \\ -R(\cos(\Delta\theta + \theta) - \cos(\theta)) \\ \Delta\theta \end{bmatrix}$$

Turning: $\Omega = (\omega_l r_l) / R_l = (\omega_r r_r) / R_r$, $R = v / \Omega$, $\Delta\theta = \Omega \Delta t$

Discretized: $\mathbf{x}_k = \mathbf{x}_{k-1} + \mathbf{b}_i$ with $i \in \{s, t\}$, respectively

Swedish Wheels: Have 3 DoF, typ. (3 pcs) equally spaced on circle.

Typ. Param. each: α, β, γ (z, x, y axes, oriented along x, z up)

Wheel rot speed from movement Typ. comp: $J \cdot \mathbf{R}(\theta) \cdot \mathbf{x} = \dot{\varphi}$, with $\mathbf{R}(\theta) \cdot \mathbf{x}$ transform from base frame to robot frame.

3 Sensors & Actuators

Meas. Model: $\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{v} + \mathbf{o}$, with $\mathbf{h}(\mathbf{x})$ determ. mean, $\mathbf{v}$ zero-mean noise, $\mathbf{o}$ unmodelled effects, $\mathbf{x}$ true state. $\mathbf{h}(\mathbf{x})$ describes how to compute $\mathbf{x}$ from known values (e.g. Intercept-Theorem). Probabilistic M.M: use random variable in definition, and s. 1.4

Motor encoders Typ. 64-2048 increments per rev; Estimate rot

Rolling-Shutter Most CMOS sensors don't take full image at once, need time stamp for each row

3.1 GNSS

Need ultra-precise time sync ($c \approx 0.3 \text{ m/ns}$). **Errors**

- Multipath problem (signal bounce) (0.5 - 100m)
- Ionosphere delays (10m)
- Satellite pos. err, trop. delay (1m)

3.2 Actuators

Hydraulic acc., easy control, power; maint., speed, price

Pneumatic price, shock abs., speed; acc., loud, maint.

3.2.1 DC Motor

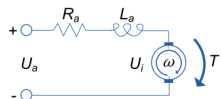
(Kirchoff) $U_a = L_a \dot{I}_a + R_a I_a + U_i$

(Torque, Lorentz Force) $T = k_T I_a$

(Induced V, Faraday) $U_i = k_i \omega$

(Mech. pow. = electric power)

$U_i I_a = k_i \omega I_a = T \omega = k_T I_a \omega \Rightarrow k_i = k_T =: k$



3.3 Cameras

Def Pinhole projection $\begin{bmatrix} u & v \end{bmatrix}^\top = \frac{f}{z} \begin{bmatrix} x & y \end{bmatrix}^\top$ with f the distance to the lens and z the distance from object $u = c_u + f \cdot x'$ and $v = c_v + f \cdot y'$ where $x' = t_x \div t_z$ and $y' = t_y \div t_z$ where u, v are the pixel x, y coords, $\mathbf{c} = [c_u, c_v]^\top$ is optical centre of cam in pixel coords, f scale factor.

The full proj: $\mathbf{u} = \begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} f & 0 & c_u \\ 0 & f & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \mathbf{K} \mathbf{c} \mathbf{t}_P$

If p. in diff frame, e.g. W -frame then $\mathbf{u} = \mathbf{K}[\mathbf{R}_{CW} \mathbf{c} \mathbf{t}_{CW}]_W \mathbf{t}_P$

3.3.1 Pinhole Camera Projection with distortion

Def Model: $\mathbf{u} = \mathbf{k}(\mathbf{d}(\mathbf{p}(\mathbf{c} \mathbf{t}_P)))$, with ($\mathbf{c}$ as above):

(Projection) $\mathbf{x}' = \mathbf{p}(\mathbf{c} \mathbf{t}_P) = t_z^{-1} \cdot [t_x, t_y]^\top$

(Distortion model) $r^2 = x'^2 + y'^2$, $\mathbf{x}'' = \mathbf{d}(\mathbf{x}')$

$$\mathbf{x}'' = \frac{1+k_1 r^2+k_2 r^4+k_3 r^6}{1+k_4 r^2+k_5 r^4+k_6 r^6} \mathbf{x}'; \quad \mathbf{x}' = \begin{bmatrix} 2p_1 x' y' + p_2 (r^2 + 2x'^2) \\ 2p_1 x' y' + p_2 (r^2 + 2y'^2) \end{bmatrix}$$

(Scale and Centre) $\mathbf{u} = \mathbf{k}(\mathbf{x}'')$ = diag($[f_u, f_v]$) $\cdot \mathbf{x}'' + \mathbf{c}$

All with k_i radial distortion params, optional for $i > 2$, p_i tang. dist. param, f_u, f_v focal length in pixels

Inverse $\mathbf{c} \mathbf{r} = [\mathbf{d}^{-1}(\mathbf{k}^{-1}(\mathbf{u})), 1]^\top$

(To unit plane) $\mathbf{x}'' = \mathbf{k}^{-1}(\mathbf{u}) = [f_u^{-1}, f_v^{-1}]^\top (\mathbf{u} - \mathbf{c})$

(Un-distort) $\mathbf{x}' = \mathbf{d}^{-1}(\mathbf{x}'')$ (usually comp. numerically)

(Compute ray) $\mathbf{c} \mathbf{r} = [\mathbf{x}', 1]^\top$

3.3.2 Undistorting a whole image

$\mathbf{u}_i = \mathbf{k}(\mathbf{d}(\mathbf{k}_{\text{new}}^{-1}(\mathbf{u}_{i,\text{new}})))$ where $\mathbf{u}_{i,\text{new}}$ is the place of pixel in output, $\mathbf{u}_i$ is the input

Omnidir. Cam undistortion model with $f(u, v) = \sum_{i=0}^N a_i \rho^i$ with $\rho = \sqrt{(u - c_u)^2 + (v - c_v)^2}$, $N = 4$ accurately describes it for most fisheye and catadioptric cameras

3.4 Depth and Range sensing

Typ. derive D.M. with sensor dir param (as vec line eq)

3.4.1 Triangulation-based

Struct. Light Single cam, single projector: **Spatial acc, no worky in bright light, interference with other IR depth cams**

Active Stereo 2 cams, 1 proj: **worky in bright light, need stereo matching, less accurate, error grows with distance**

3.4.2 Classic Stereo

Both images: same plane, focal length f , centre, x -axis. Given corresponding pixels $[u_l, v]$ and $[u_r, v]$: disparity (pixel offset) $d = u_r - u_l$; $z = \frac{b \cdot f}{d}$, then projected into left (right) cam, $u_l = f \cdot \frac{x}{z} + c_u$ or $u_r = f \cdot \frac{x-b}{z} + c_u$, b distance between cameras. Then often apply pinhole camera projection (to e.g. get 3D point coords, comp $\mathbf{c} \mathbf{r}$, then point $\mathbf{z} \cdot \mathbf{c} \mathbf{r}$). Uncertainty introduced with 1.-order error propag. $\Delta z = \frac{\partial}{\partial d} z \Delta d$, with Δd error of d

3.4.3 Time of Flight, Projection

No occlusions/shadows, Interference with other dev, multipath leading to larger distances sensed

Proj. $\mathbf{z} = \begin{bmatrix} u \\ d \end{bmatrix} = \begin{bmatrix} \mathbf{k}(\mathbf{d}(\mathbf{p}(\mathbf{c} \mathbf{t}_P))) \\ [0, 0, 1]_c \mathbf{t}_P \end{bmatrix}$ **Back:** $\mathbf{c} \mathbf{t}_P = \begin{bmatrix} d \mathbf{x}' \\ d \end{bmatrix}$ This $\mathbf{x}'$ is obtained from pinhole cam projection.

3.4.4 Range Sensors

Ultrasonic Typ. freq: 40kHz - 180kHz, Range: 12cm - 5m, Acc: $\approx 2\text{cm}$, rel error $\approx 2\%$ **meas. for transp. surf., cheap(ish), Cone wider, reflect. angle dep, wind / currents**

LiDAR Time-of-Flight-based, **Accuracy, range, works in bright light, Complex, expensive, one timestamp per measurement.** Typical Ranges: up to 100m

4 Multi-Sensor Estimation

4.1 Linearization

$f(\mathbf{x}) \approx f(\bar{\mathbf{x}}) + \mathbf{J}_f|_{\mathbf{x}=\bar{\mathbf{x}}}(\mathbf{x} - \bar{\mathbf{x}})$, f' , no vec in 1D; $\bar{\mathbf{x}}$ lin. p.

Def Jac. $\mathbf{J}_f$ rows for eq of f ; cols for vars of each eq.

Part. diff; Approx. using finite differences $\frac{f(\bar{\mathbf{x}}+h) - f(\bar{\mathbf{x}})}{h}$, or central differences (vector of $\frac{f(\bar{\mathbf{x}}+h_i \mathbf{e}_i) - f(\bar{\mathbf{x}})}{h_i}$, with $\mathbf{e}_i$ unit vec)

4.2 Linear Least Squares

Goal: $\text{argmin}_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A} \mathbf{x} - \mathbf{b}\|_2^2$, $\mathbf{A}$: rows i -th datap. col c : t_i^{c-1} .

Man. sol.: comp. $\mathbf{M} = \mathbf{A}^\top \mathbf{A}$, $\mathbf{b}' = \mathbf{A}^\top \mathbf{b}$, then $\mathbf{M} \mathbf{x} = \mathbf{b}'$.

Prob. sol.: $\text{argmax} \mathbb{P}(\mathbf{x} | \mathbf{z})$ with Maximum ...

- Likelihood** $\mathbb{P}(\mathbf{x} | \mathbf{z}) \propto \mathbb{P}(\mathbf{z} | \mathbf{x}) = \prod_{i=1}^N \mathbb{P}(\mathbf{z}_i | \mathbf{x})$
- a Post** $\mathbb{P}(\mathbf{x} | \mathbf{z}) \propto \mathbb{P}(\mathbf{z} | \mathbf{x}) \mathbb{P}(\mathbf{x}) = \mathbb{P}(\mathbf{x}) \prod_{i=1}^N \mathbb{P}(\mathbf{z}_i | \mathbf{x})$

4.3 Non-Linear Least Squares

Find $\mathbf{x}^* = \text{argmax} \mathbb{P}(\mathbf{x} | \mathbf{z}) = \text{argmin}(-\log(\mathbb{P}(\mathbf{x} | \mathbf{z})))$

Gauss-Newton Need func F and its Jacobian (or derivative) D

```
def gauss_newton(x: np.ndarray, F, DF, tol=1e-6):
    s = np.linalg.lstsq(DF(x), F(x))[0] # least sq.
    x = x-s; k = 1
    while np.linalg.norm(s) > tol * np.linalg.norm(x):
        s = np.linalg.lstsq(DF(x), F(x))[0]
        x = x-s; k += 1 # k opt for max iter
    return x, k
```

Levenberg-Marquardt Minimize $\|F(x^{(k)}) + DF(x^{(k)})\| + \lambda \|s\|$.

Local Param.

4.4 Bayes Filter (in DAG)

$\mathbf{x}_k^R$ state at time k , $\mathbf{z}_k^p$ dist. meas., $\mathbf{u}_k^p$ wheel odometry (= meas.). Typ. care ab. curr. state: altern. pred. & update.

4.5 Particle Filter

Is a bayes filter approximating the state distribution with a set of random samples. Update step:

- Apply Bayes rule $w'_{k,s} = \mathbb{P}[\mathbf{z}_k | \mathbf{x}_{k,s}] w_{k-1,s}$
- Renormalize: $w_{k,s} = w'_{k,s} \div \sum_s w'_{k,s}$
- Resample: rand. sel. S particles acc. to weights and $w_{k,s} = S^{-1}$

4.6 Kalman Filter (KF)

Bayes Filter for Gauss. dist of R.V. & linear meas. model. Initial state $\mathbf{x}_0 \sim \mathcal{N}(\hat{\mathbf{x}}, \mathbf{P}_0)$, $\mathbf{P}_0$ previous covariance;

Prediction With linear state transition model ($\mathbf{u}_k$ odometry, $\mathbf{w}_k$ noise (covariance $\mathbf{Q}_k$)):

$\mathbf{x}_k = \mathbf{F} \mathbf{x}_{k-1} + \mathbf{G} \mathbf{u}_k + \mathbf{L} \mathbf{w}_k$ with $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$:

- **Mean** $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1} + \mathbf{G}\mathbf{u}_k$
- **Covariance** $\mathbf{P}_{k|k-1} = \mathbf{F}\mathbf{P}_{k-1|k-1}\mathbf{F}^\top + \mathbf{L}\mathbf{Q}_k\mathbf{L}^\top$

Update Lin. meas.: $\tilde{\mathbf{z}}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k$ with $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$:

- **Measurement residual**: $\mathbf{y}_k = \tilde{\mathbf{z}}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}$
- **Residual Covariance**: $\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k$
- **Kalman gain**: $\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^\top\mathbf{S}_k^{-1}$
- **Updated mean**: $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\mathbf{y}_k$
- **Updated Cov.**: $\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1}$

4.7 Extended Kalman Filter (EKF)

Non-l. state trans. model $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)$ as above:

- **Mean**: $\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$
- **Covariance**: $\mathbf{P}_{k|k-1} = \mathbf{F}_k\mathbf{P}_{k-1|k-1}\mathbf{F}_k^\top + \mathbf{L}_k\mathbf{Q}_k\mathbf{L}_k^\top$
With $\mathbf{F}_k$ linearisation $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ and $\mathbf{L}_k$ lin. $\frac{\partial \mathbf{f}}{\partial \mathbf{w}}$

Update Non-Linear meas.: $\tilde{\mathbf{z}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k$:

- **Meas. residual**: $\mathbf{y}_k = \tilde{\mathbf{z}}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1})$

Difference to KF: $\mathbf{H}$ becomes $\mathbf{H}_k$, and $\mathbf{H}^\top$ is $\mathbf{H}_k^\top$

5 SLAM to Spatial AI

5.1 Keypoints

Corner det. $SSD(\Delta x, \Delta y) \approx [\Delta x \ \Delta y] \mathbf{M} [\Delta x \ \Delta y]^\top$ with $\mathbf{M} = \mathbf{R}^\top \text{diag}(\lambda_1, \lambda_2) \mathbf{R}$; λ_i E.V. of $\mathbf{M}$; κ const 0.04-0.15;
 $\mathbf{R} = \text{det}(\mathbf{M}) - \kappa \cdot \text{TR}(\mathbf{M})^2 = \lambda_1 \lambda_2 - \kappa(\lambda_1 + \lambda_2)^2$;

$$\mathbf{M} = \sum_{x,y \in P} \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

Blob Detection (I is the image)

Laplacian of Gaussian (LoG): $L = g(x, y, t) \cdot I(x, y)$. Then apply

$$\text{Laplacian Operator } \nabla_{\text{norm}}^2 L = t \left(\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} \right)$$

Diff. of Gaussians (DoG): $\Delta L = L(x, y, t) - L(x, y, kt)$

SIFT Detector (1) Subsample + Blur **(2)** DoG on each res. image
(3) Keypoints extrema in DoG pyramid

5.2 Bootstrapping

PnP Problem Persp. n -P. Find sol. for camera pose *directly*

RANSAC RANdom SAmpling Consensus for find. outliers & correct

Stereo Triang. Given two rays (known poses for points in 2D). Find good point in 3D. Fast sol: **Midpoint Method**:

1 Find p. along ray w/ min. dist (Lin. Least Squares)

$$\lambda = [\lambda_1 \ \lambda_2]^\top = \text{argmin} \| (w\mathbf{t}_{C_2} + \lambda_2 w\mathbf{e}_2) - (w\mathbf{t}_{C_1} + \lambda_1 w\mathbf{e}_2) \|^2$$

2 Solve normal equation $\mathbf{A}\lambda = \mathbf{b}$ with $\mathbf{q} = -w\mathbf{e}_1^\top w\mathbf{e}_2$:

$$\mathbf{A} = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{e}_1^\top \cdot (w\mathbf{t}_{C_2} - w\mathbf{t}_{C_1}) \\ -\mathbf{e}_2^\top \cdot (w\mathbf{t}_{C_2} - w\mathbf{t}_{C_1}) \end{bmatrix} \quad C_i \text{ cam}$$

3 Pick midp. $w\mathbf{t}_P = 0.5(\tau_1 + \tau_2)$; $\tau_n = w\mathbf{t}_{C_n} + \lambda_n w\mathbf{e}_n$

5.3 Place Recognition

Idea: Build vocab from “visual words” (in Training). **Runtime** Detect keypoints; Extract descriptors; Build histogram; Query DB for similarity; If no match, insert into DB, else use to compute with e.g. RANSAC

5.4 Mapping

Problem Formulations Localisation (always static given map):

$\mathbf{x}_{R,k}^* = \text{argmax} \mathbb{P}(\mathbf{x}_{R,k} | \mathbf{x}_M, \mathbf{z}_{1:k}, \mathbf{u}_{1:k})$ (recursive)

$\mathbf{x}_{R,1:k}^* = \text{argmax} \mathbb{P}(\mathbf{x}_{R,1:k} | \mathbf{x}_M, \mathbf{z}_{1:k}, \mathbf{u}_{1:k})$ (batch)

SLAM $\{\mathbf{x}_{R,k}^*, \mathbf{x}_M^*\} = \text{argmax} \mathbb{P}(\mathbf{x}_R, \mathbf{x}_M | \mathbf{z}_{1:k}, \mathbf{u}_{1:k})$ (as $\uparrow$)

Mapp.: $\mathbf{x}_M^* = \text{argmax} \mathbb{P}(\mathbf{x}_M | \mathbf{x}_{R,1:k}^*, \mathbf{z}_{1:k}, \mathbf{u}_{1:k})$ with given poses $\mathbf{x}_{R,1:k}^*$. Thus temp. model $\mathbf{u}_{1:k}$ doesn't matter.

Prob. Occ. Grid $\mathbb{P}(f_j) = \mathbb{P}(\neg o_j) = 1 - \mathbb{P}(o_j)$, with $\mathbb{P}(o_j)$ prob. cell j occupied; pairwise independent.

Occ. Map. w/ depth sensor $\mathbb{P}(o_j | \mathbf{x}_{R,1:k}) =$

$$\frac{\mathbb{P}(o_j | \mathbf{z}_{R,k}, \mathbf{z}_k) \mathbb{P}(\mathbf{z}_k | \mathbf{x}_{R,k}) \mathbb{P}(o_j | \mathbf{z}_{R,1:k-1}, \mathbf{z}_{1:k-1})}{\mathbb{P}(o_j) \mathbb{P}(\mathbf{z}_k | \mathbf{x}_{R,1:k}, \mathbf{z}_{1:k-1})}$$

map prior; Prev. occ. est.; Occ. based on curr. range meas.;

Update function: $l(a) := \log(\text{Odds}(a))$ with $\text{Odds}(a) = \frac{\mathbb{P}(a)}{1-\mathbb{P}(a)}$
(inv. sensor model): $l(o_j | \mathbf{x}_{R,1:k}, \mathbf{z}_{1:k}) =$

$$l(o_j | \mathbf{x}_{R,k}, \mathbf{z}_k) + l(o_j | \mathbf{x}_{R,1:k-1}, \mathbf{z}_{1:k-1}) - l(o_j)$$

In 3D 3D voxel j as signed dist. s and weight w , update: $s_k = \frac{w_{k-1}s_{k-1} + s_k}{w_{k-1} + 1}$ with $w_k = \min(w_{\max}, w_{k-1} + 1)$

Impl. Using HashTables or octree

5.5 Dense Tracking and Loop Closure

Idea: Min. sum of sq. errors over all pixels w/ Gauss-Newton.

6 Planning & Control

6.1 SISO & MIMO

Single-Input-Single-Output: r Reference (e.g. speed), u Sys. Input (e.g. gas pedal pos), z Sys. Out. (e.g. speed of car)

Multiple-Input-Multiple-Output: r Reference (typ: trajectory), u Sys. Input (e.g. 4 rotor speeds), x internal states (pos, orient, speed, rot. speed), z Sys. Output (e.g. speed of car)

6.2 Proportional-Integral-Differential (PID)

$u(t) = k_p e(t) + k_i \int_{t_0}^t e(\tau) d\tau + k_d \frac{de(t)}{dt}$, where params k_p (curr), k_i (long-term), k_d (trend) reduce corresp. errors

Drone Control ${}_B\mathbf{F}$ and ${}_B\mathbf{M}$ as in sect. 2.6, use $\mathbf{u}' = {}_B\mathbf{M}$.

6.3 Linear Quadratic Regulator (LQR)

For lin. cont.-time dyn. $\dot{\mathbf{x}}(t) = \mathbf{F}_c \mathbf{x}(t) + \mathbf{G}_c \mathbf{u}(t)$.

We try to minimise cost functional (for $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$)

$$J = \int_t^\infty \mathbf{x}(\tau)^\top \mathbf{Q} \mathbf{x}(\tau) + \mathbf{u}(\tau)^\top \mathbf{R} \mathbf{u}(\tau) d\tau$$

Solution: $\mathbf{K} = \mathbf{R}^{-1}(\mathbf{G}_c^\top \mathbf{P})$, with $\mathbf{P}$ found from

$$\mathbf{F}_c^\top \mathbf{P} + \mathbf{P} \mathbf{F}_c - (\mathbf{P} \mathbf{G}_c) \mathbf{R}^{-1} (\mathbf{G}_c^\top \mathbf{P}) + \mathbf{Q} = \mathbf{0}$$

Finding $\mathbf{K}$ is expensive, but *offline*, at runtime only $\mathbf{u}(t)$.

Non-Lin: Approx, $\delta \dot{\mathbf{x}}(t) = \mathbf{F}_c \delta \mathbf{x}(t) + \mathbf{G}_c \delta \mathbf{u}(t)$

6.4 MPC

Cost function ($p(\mathbf{x}_N)$ terminal cost, sum the stage cost)

$$J_{0 \rightarrow N}(\mathbf{x}_0, \mathbf{u}_0, \dots, \mathbf{u}_{N-1}) = p(\mathbf{x}_N) + \sum_{k=0}^{N-1} q(\mathbf{x}_k, \mathbf{u}_k)$$

We minimize the above s.t. for $k \in \{0, \dots, N-1\}$ we have $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$, $\mathbf{g}(\mathbf{x}_k, \mathbf{u}_k) \leq \mathbf{0}$ and $\mathbf{x}_N \in \mathcal{X}_f$ and $\mathbf{x}_0 = \mathbf{x}(0)$

Finite-Horizon Lin-Quad Control Quad. Cost:

$$J_{0 \rightarrow N}(\mathbf{x}_0, \mathbf{u}_0, \dots) = \mathbf{x}_N^\top \mathbf{P} \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k$$

without constraints, **State Feedback Law** $\mathbf{u}_0^* = -\mathbf{K}\mathbf{x}(0)$. With constraints, minimize as above, $\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k$

6.5 Motion Planning

Config. Space: $\mathcal{C}$. Subset of robot states for planning.

Free C-Space: $\mathcal{C}_{\text{free}}$, **C-Space Obstacles** $\mathcal{C}_{\text{obst}}$ (occupied)

Collision checker: $c(x) : \mathcal{C} \rightarrow \{0, 1\}$

Visibility graph: Connect corners, goal outside obstacles

Voronoi Diagram: Edges at max. dist. from obst. (benefit: safer paths). Also tends to be faster than Dijkstra.

6.5.1 A* Algorithm

Function $h(\dots)$ is a lower bound of optimal cost.

```

1 procedure ASTAR(Graph, start, goal)
2   for each v in Graph.Verticies do
3     dist[v] ← ∞
4     totDistEst[v] ← ∞
5     prev[v] ← undef
6   insert start into openSet
7   dist[start] ← 0
8   totDistEst[start] ← h(start, goal)
9   while openSet not empty do
10    u ← vert in openSet w/ min totDistEst
11    remove u from openSet
12    if u == goal then
13      return dist[u], prev
14    for each neighbour v of u do
15      alt ← dist[u] + G.Edges(u, v).dist
16      if alt < dist[v] then
17        dist[v] ← alt
18        totDistEst[v] ← alt + h(v, goal)
19        insert v into openSet
20        prev[v] ← u
21  return ∞, prev

```

6.5.2 Rapidly-Exploring Random Tree (RRT)

```

1 procedure RRT(start, goal)
2   INSERTVERTEX(start, Graph)
3   for k < MAX_ITER do
4     s ← scal.unif.rand.sample on [0, 1]
5     if s < GOAL.CONNECT.PROB then
6       x ← goal
7     else
8       x ← random coll.-free config
9     x_n ← NEARESTCONFIGURATION(Graph, x)
10    x_f ← STOPPINGCONFIGURATION(x_n, x)
11    if x_f ≠ x_n then
12      insertVertexGraph, x_f
13      ▷ Extension of RRT* goes here
14      insertEdgeGraph, x_n, x_f
15    if x_f == x_n then
16      return SUCCESS, Graph
17  return FAILURE, Graph

```

Returns a collision-free path as graph. Need nearest neighbour search. Extension to RRT* to make path better:

```

1  $X_{\text{near}} \leftarrow \text{NEIGHBOURS}(\text{Graph}, x_f, R)$ 
2  $x_{\text{min}} \leftarrow \text{NEIGHBOURS}(\text{Graph}, x_f, R)$ 
3  $c_{\text{min}} \leftarrow \text{COST}(x_n) + \text{EDGECOST}(x_n, x_f)$ 
4 for each  $x_{\text{near}}$  in  $X_{\text{near}}$  do
5   if  $\text{EDGECOLLISIONFREE}(x_{\text{near}}, x_f)$  and
      $\text{COST}(x_{\text{near}}) + \text{EDGECOST}(x_{\text{near}}, x_f) < c_{\text{min}}$  then
6      $\text{INSERTEDGE}(\text{Graph}, x_f, x_{\text{near}})$ 
7      $x_{\text{parent}} \leftarrow \text{PARENT}(\text{Graph}, x_{\text{near}})$ 
8      $\text{REMOVEEDGE}(\text{Graph}, x_{\text{parent}}, x_{\text{near}})$ 

```

Informed RRT* extension: Once conn. betw. start and goal found, restrict sampling to (hyper)ellipsoid.

6.5.3 Exploration

Find action sequence by finding goals, planning collision-free paths there, choose goal/path with maximum utility (typically max map information)

6.5.4 Collision Avoidance

Dynamic Window Approach Assume: Robot moves inst. on circ. arcs (v, ω) . Compute arcs with coll. **Accounts for Kino-Dyn, Cost func prone to loc. min, assumes static obj**

Vel. Obst. Assume: Robot in str. line (v_x, v_y) . Comp pos with coll. **Vel. of obj, prone to local optima, no kino-dyn.**

Potential Field Methods Define *repulsive* and *attractive* potential $c = c_{\text{att}} + c_{\text{rep}}$. With e.g. $c_{\text{att}} = \frac{1}{2} k_{\text{att}} \|\mathbf{x} - \mathbf{x}_{\text{goal}}\|^2$ and $c_{\text{rep}} =$

$$\begin{cases} \frac{1}{2} k_{\text{rep}} \left(\frac{1}{\rho(\mathbf{x})} + \frac{1}{\rho_{\text{lim}}} \right) & \rho \leq \rho_{\text{lim}} \\ 0 & \text{else} \end{cases}$$

Simple control laws, may trap in loc. min., no diff. const, no guar. to avoid coll

6.6 Learning To Act

Used if there is no state-transition model or cost/reward func.

6.6.1 Markov Decision Process

The goal is to maximize the reward. Along the route, get small reward, at the end large reward (good or bad).

Def. by states $\mathbf{x} \in \mathcal{X}$ (RL: s), actions $\mathbf{u} \in \mathcal{U}$ (RL: a), prob. state trans. $\mathcal{T}(\mathbf{x}, \mathbf{u}, \mathbf{x}_+) = \mathbb{P}(\mathbf{x}_+ | \mathbf{x}, \mathbf{u})$, reward func $\mathcal{R}(\mathbf{x}, \mathbf{u}, \mathbf{x}_+)$, start state $\mathbf{x}_0$, optional terminal state $\mathbf{x}_N$.

Utility Func Expected reward: $V = \sum_{k=0}^N r_k$ or *discounted* reward $V = \sum_{k=0}^{\infty} \gamma^k r_k$ with $\gamma < 1$ (if inf. runtime)

Solving (Val iter) $V_0(\mathbf{x}) = 0$ and $V_{i+1}(\mathbf{x}) = \max_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$ with

$$Q(\mathbf{x}, \mathbf{u}) = \sum_{\mathbf{x}_+} \mathbb{P}(\mathbf{x}_+ | \mathbf{x}, \mathbf{u}) [\mathcal{R}(\mathbf{x}, \mathbf{u}, \mathbf{x}_+) + \gamma V_i(\mathbf{x}_+)]$$

Repeat until conv. to V^* ($\mathcal{O}(|\mathcal{U}||\mathcal{X}|^2)$ per iter). Optimal policy: $\pi^*(\mathbf{x}) = \operatorname{argmax}_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$

Using policy iter:

```

1 Choose  $\pi_0(\mathbf{x})$ 
2 while policy has not converged do
3   repeat  $V_{i+1}^{\pi_j}(\mathbf{x}) = Q(\mathbf{x}, \pi(\mathbf{x})) \forall \mathbf{x}$  and fixed pol.  $\pi_j$ 
4   until values converge
5 One step:  $\pi_{j+1}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$  with  $V_i = V_{i+1}^{\pi_j}$ 

```

Model-based learning uses empirical models of $\mathcal{T}$ and $\mathcal{R}$

6.6.2 Reinforcement Learning

Passive Direct Evaluation Act according to policy π , store sum of discounted rewards, average them. (But too simple)

Sample-Based Use $V_{i+1}^{\pi}(\mathbf{x}) = Q(\mathbf{x}, \pi(\mathbf{x}))$ w/ $V_0^{\pi}(\mathbf{x}) = 0$. We need state trans. model, instead $\tilde{R}_j$ (approx. prob. w/ statistics) and thus

$$V_{i+1}^{\pi}(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^N \tilde{R}_j(\mathbf{x}, \pi(\mathbf{x}), \mathbf{x}_+) + \gamma V_i^{\pi}(\mathbf{x}_+)$$

Active Find optimal policy π instead of state values $V(\mathbf{x})$.

Q-Learn. Here: restate Val. Iter in Q -Values ($Q_0(\mathbf{x}, \mathbf{u}) = 0$):

$$Q_{i+1}(\mathbf{x}, \mathbf{u}) = Q(\mathbf{x}, \mathbf{u}) \quad \text{with } V_i(\mathbf{x}_+) = \max_{\mathbf{u}} (Q_i(\mathbf{x}_+, \mathbf{u}))$$

Compute using sample:

$$\tilde{Q}_i(\mathbf{x}, \mathbf{u}) = \tilde{R}_i(\mathbf{x}, \mathbf{u}, \mathbf{x}_+) + \gamma \max_{\mathbf{u}} (Q_i(\mathbf{x}_+, \mathbf{u}))$$

Then update: $Q_{i+1}(\mathbf{x}, \mathbf{u}) = (1 - \alpha) Q_i(\mathbf{x}, \mathbf{u}) + \alpha \tilde{Q}_i(\mathbf{x}, \mathbf{u})$. Called off-policy learning, needs exploration. Simplest is random actions (ε -greedy): ε is prob. to act randomly, $1 - \varepsilon$ is prob. to act on pol.

Space explored, still doing random stuff

Approaches *Model-based* (estimate trans. model, e.g. Dyna), *Value-based* (estimate val or Q -func and extract pol., e.g. Q-Learn), *Actor-Critic* (estim. val or Q of curr. pol., improve pol., e.g. A3C, SAC), *Policy-Gradient* (diff. expect. reward w.r.t. params of policy network, e.g. REINFORCE)